A second-order semi-implicit δf method for hybrid simulation

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1. Introduction

Numerical simulation has played an increasingly important role in studying plasma phenomena including both magnetic confinement fusion and magnetic reconnection in space and laboratory plasmas. According to the physical models employed, they are often termed as Magnetohydrodynamics (MHD), two-fluid and Particle-In-Cell (PIC) simulation. MHD and two-fluid theory do not include kinetic effects. On the other hand, full kinetic simulation has all the relevant physics but requires resolving multiple time and spatial scales over many orders of magnitude, which imposes difficulty in numerical simulations, especially when modeling low frequency phenomena. As a result, full kinetic simulation usually uses a relatively small simulation box size and/or ion-to-electron mass ratio. For modeling low frequency phenomena, one plausible approach is to include kinetic ions within an MHD model [1–14]. In one kinetic MHD hybrid approach [1,2], a multi-fluid model is closed with a pressure tensor calculated from the particle distribution function, while the ion density and flow velocity are calculated through continuity equation and ion momentum equation. Another approach, the so-called “current closure”, uses Lorentz force ions and fluid electrons to calculate the ion density and ion current density directly from the ion distribution function governed by the Vlasov equation [3–14]. The electric and magnetic fields are calculated from the generalized Ohm’s law (electron momentum equation) and Faraday’s law. Some good reviews of Hybrid methods can be found in [15,16].

In this paper, we present a “current closure” hybrid model with Lorentz force ions and fluid electrons similar to Harned [7]. When quasi-neutrality is assumed, the electron density and flow needed in the generalized Ohm’s law can be calculated
by the ion density and ion flow obtained directly from the particles. Our model includes the electron inertia effect. The particles are treated with the $\delta f$ method [17], which enables us to obtain clean linear stages of various instabilities. The $\delta f$ method also sets a limit that the code will not work with very large perturbations. Based on previous efforts [18,19], we have developed a second-order accurate semi-implicit algorithm to eliminate the constraint on the timestep imposed by the fast compressional wave. The code has been developed by coding within the three-dimensional (3D) general geometry toroidal gyrokinetic code GEM [20,21]. Since we use real space to avoid Fourier convolutions due to the inhomogeneous equilibrium fields, the field solver involves large matrix inversion. Assuming a grid of $N1 \times N2 \times N3$, for one-dimensional (1D) inhomogeneous equilibrium in $x$ direction, the matrix size is $3N1 \times 3N1$. To accommodate the inhomogeneities of the equilibrium in the other two directions, the matrix size will be $27N1N2N3 \times 27N1N2N3$ which may be extremely large. For demonstration of the code, all simulations presented in this paper assume a slab geometry with a 1D inhomogeneous equilibrium.

We have benchmarked the simulation using linear Alfvén waves, whistler waves and ion acoustic waves including ion Landau damping. In particular, we have performed simulations of the nonlinear Landau damping of an ion acoustic wave, which to our knowledge has not been reported in the literature. We have studied the density damping associated with the implicit time advance using a numerical dispersion relation for whistler waves. The code has been tested nonlinearly by simulating the full evolution of the resistive tearing mode. The linear growth rate and eigenmode structure show reasonable agreement with the resistive MHD theory [22,23]. Important nonlinear tearing mode dynamics such as the Rutherford stage and saturation are also captured. The Rutherford growth rate and saturation island width are consistent with previous MHD studies [24–29].

The paper is organized as follows. The governing equations are described in Section 2. In Section 3, the second-order accurate semi-implicit algorithm is presented in combination with the $\delta f$ method. Section 4 presents benchmarks of Alfvén waves, ion sound waves and whistler waves with linear theory. Section 5 investigates the resistive tearing mode. Linear growth rates and eigenmode structure are compared with resistive MHD theory. Systematic studies of important nonlinear tearing mode dynamics such as the Rutherford stage and saturation are also reported.

2. Equations

In this section, we present the governing equations, including the ion and electron models and the field equations. The ions follow the equations of motion

$$\frac{dx_i}{dt} = v_i,$$

(1)

$$\frac{dv_i}{dt} = \frac{q_i}{m_i}(E + v_i \times B) - \frac{q_i}{m_i} \eta j,$$

(2)

where $\eta$ is the resistivity and $j$ is the current density. The ion charge and mass are noted as $q_i$ and $m_i$. The third term on the right-hand-side (rhs) of Eq. (2) provides the momentum balance. The field equations are Ampere’s law and Faraday’s law

$$\nabla \times B = \mu_0 j = \mu_0 (q_i n_i u_i - e ne u_e),$$

(3)

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$  

(4)

Here, $u_i$ and $u_e$ stands for the ion and electron flow velocity. Computationally, it is convenient to solve for the electric field from the generalized Ohm’s law that arises from the electron momentum equation

$$en_e (E + u_e \times B) = en_e \eta j - \nabla \cdot \Pi_e - m_e \frac{\partial (n_e u_e)}{\partial t},$$

(5)

where electron stress tensor is $\Pi_e = \int f_e m_e v dv$, and it is determined by the selection of electron models. Upon substituting in the Ampere’s equation $j = j_i - j_e = \nabla \times B / \mu_0$, Eq. (5) can be rewritten as

$$en_e E = -j_i \times B + \frac{1}{\mu_0} (\nabla \times B) \times B + \frac{en_e \eta}{\mu_0} (\nabla \times B) - \nabla \cdot \Pi_e - m_e \frac{\partial (n_e u_e)}{\partial t}.$$  

(6)

The last term on the rhs represents the electron inertial effect, which is calculated from Ampere’s law in combination of the ion momentum equation [30]. First, we take the time derivative of Ampere’s law, yielding

$$\mu_0 \left( q_i \frac{\partial n_i u_i}{\partial t} - e \frac{\partial n_i u_i}{\partial t} \right) = \nabla \times \frac{\partial B}{\partial t} - \nabla \times \nabla \times E.$$  

(7)

The first term on the left hand side (lhs) is obtained from the ion momentum equation

$$m_i \frac{\partial n_i u_i}{\partial t} = q_i n_i (E + u_i \times B) - \nabla \cdot \Pi_i - q_i n_i \frac{\eta}{\mu_0} \nabla \times B.$$  

(8)
where $\mathbf{\Pi}_i = f m_i \mathbf{v} \mathbf{w} \mathbf{v}$ is the ion stress tensor. With Eqs. (7) and (8), the electron inertia term takes the form

$$m_e \frac{\partial (n_e \mathbf{v}_e)}{\partial t} = m_e \frac{q_e}{m_i} \left( q_n n_i (E + \mathbf{u}_i \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_e - q_e n_e \frac{\eta_T}{\mu_0} \nabla \times \mathbf{B} \right) + \frac{m_e}{\mu_0 e} \nabla \times (\nabla \times \mathbf{E}).$$

(9)

Using Eq. (9) and quasi-neutrality $n_i = n_e$, the generalized Ohm's law Eq. (6) becomes

$$en_i \left( 1 + \frac{m_e}{m_i} \frac{q_i^2}{e^2} \right) \mathbf{E} + \frac{m_e}{\mu_0 e} \nabla \times (\nabla \times \mathbf{E}) = - \left( 1 + \frac{m_e}{m_i} \frac{q_i^2}{e^2} \right) \mathbf{j}_i \times \mathbf{B} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \eta \frac{en_i}{\mu_0} \left( 1 + \frac{m_e}{m_i} \frac{q_i^2}{e^2} \right) \nabla \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}_e$$

$$+ \frac{m_e}{\mu_0 e} \nabla \times \mathbf{\Pi}_e.$$  

(10)

In general, one needs a kinetic electron model to obtain the electron stress tensor $\mathbf{\Pi}_e$. In this paper, however, the electrons are assumed isothermal and the electron pressure tensor reduces to

$$P_e = n_e T_e = n_e T_e,$$

(11)

where $T_e$ is the constant electron temperature. Therefore, the above equations are closed with ion density and current density obtained directly from the particle ions. Drift kinetic electrons will be addressed in future work and have been tested already within the framework of a first-order accurate implicit method [19].

3. Numerical method

We utilize the usual nonlinear $\delta f$ method [17], which is effective in reducing particle noise to obtain clean linear results with relatively small number of particles. The $\delta f$ method originates from the Klimontovich representation of the particle distribution function

$$f(\mathbf{x}, \mathbf{v}, t) = \sum_j \delta(\mathbf{x} - \mathbf{x}_j(t)) \delta(\mathbf{v} - \mathbf{v}_j(t)),$$

(12)

which is a natural solution to the Vlasov equation. If the distribution function $f$ can be divided into an equilibrium part $f_0$ and a perturbed part $\delta f$ as $f = f_0 + \delta f$, a particle weight can be assigned to each particle $w_j \approx \delta f / f_0$. And $\delta f$ is then represented as

$$\delta f = \sum_j w_j \delta(\mathbf{x} - \mathbf{x}_j(t)) \delta(\mathbf{v} - \mathbf{v}_j(t)).$$

(13)

The evolution of the particle weight is described by the so-called weight equation

$$\frac{d w}{d t} = - \left( \frac{f}{g} - w \right) \frac{1}{dt} \ln f_0,$$

(14)

where $g$ is the marker distribution loaded in the simulation. Usually, if the particle weight $w \ll 1$, the above equation can be simplified as

$$\frac{d w}{d t} = - \frac{1}{dt} \ln f_0.$$  

(15)

Once the particle weights are known, the perturbed particle density and current can be calculated as follows

$$\delta n = \sum_j w_j S(\mathbf{x} - \mathbf{x}_j),$$

(16)

$$\delta j = \sum_j w_j \mathbf{v}_j S(\mathbf{x} - \mathbf{x}_j),$$

(17)

where $S(\mathbf{x} - \mathbf{x}_j)$ is the shape function.

For $\rho_s$, scale instabilities in tokamaks, $k \rho_s \sim 1$, $\beta \sim 0.01$, the compressional wave frequency $\omega / \Omega_i \gg 10$, which places a severe constrain on the time step $\Omega_i \Delta t \ll 0.01$. To eliminate the strict timestep constraint, we have developed a second-order accurate semi-implicit method with an adjustable centering parameter.

The velocity is normalized to the ion sound speed $c_i = T_i / m_i$, length to the ion sound gyro-radius $\rho_s = m_i c_i / e B_0$ and time to $\Omega_i^{-1} = m_i / e B_0$. The charge and mass are normalized to that of a proton. $\rho_e = \mu_0 n_0 T_e / B_0^2$ is defined upon the uniform background plasma. We now write down the equations in numerical form. The equations of motion are

$$\frac{\mathbf{x}^{n+1} - \mathbf{x}^n}{\Delta t} = (1 - \theta) \mathbf{v}^n + \theta \mathbf{v}^{n+1},$$

(18)

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = (1 - \theta) \mathbf{a}^n + \theta \mathbf{a}^{n+1},$$

(19)
\[
\frac{\Delta t}{\Delta t} \frac{w^{n+1} - w^n}{\Delta t} = -(1 - \theta) \left( \mathbf{v} \cdot \nabla + \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}} \ln f_0(\mathbf{x}^n, \mathbf{v}) \right) - \theta \left( \mathbf{v}^{n+1} \cdot \nabla + \mathbf{a}^{n+1} \cdot \frac{\partial}{\partial \mathbf{v}} \ln f_0(\mathbf{x}^{n+1}, \mathbf{v}^{n+1}) \right),
\]
(20)
where \( \mathbf{a} = \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \eta \nabla \times \mathbf{B}/\beta_e \). If \( f_0 \) is Maxwellian and no resistivity is present, the weight equation Eq. (20) reduces to
\[
\frac{\Delta t}{\Delta t} \frac{w^{n+1} - w^n}{\Delta t} = \frac{q_i}{f_0} \left( (1 - \theta) (\mathbf{E} \cdot \mathbf{v}) + \theta (\mathbf{v}^{n+1} \cdot \mathbf{v}^{n+1}) \right),
\]
(21)
which is used in Section 4 for linear comparisons.

Faraday’s law is solved using
\[
\frac{\Delta t}{\Delta t} \frac{\delta \mathbf{B}^{n+1} - \delta \mathbf{B}^n}{\Delta t} = -[(1 - \theta) \nabla \times \mathbf{E}^n + \theta \nabla \times \mathbf{E}^{n+1}],
\]
(22)
and the generalized Ohm’s law is
\[
(n_{\theta 0} + \delta n_{\theta 1}) \left( 1 + \frac{m_e}{m_i} q_i^2 \right) \mathbf{E}^{n+1} + \frac{1}{m_i} \frac{\partial}{\partial \mathbf{v}} \nabla \times (\nabla \times \mathbf{E}^{n+1}) = - \left( 1 + \frac{m_e}{m_i} q_i^2 \right) \delta \mathbf{j}_{\theta 1}^{n+1} \times (\mathbf{B}_0 + \delta \mathbf{B}^{n+1}) + \frac{1}{\beta_e} \left( \nabla \times \delta \mathbf{B}^{n+1} \right)
\times \mathbf{B}_0 + \frac{1}{\beta_e} \left( \nabla \times (\mathbf{B}_0 + \delta \mathbf{B}^{n+1}) \right) \delta \mathbf{B}^{n+1} + \frac{\eta}{\beta_e} \left( 1 + \frac{m_e}{m_i} q_i^2 \right) (n_{\theta 0} + \delta n_{\theta 1}^{n+1}) \nabla \times \delta \mathbf{B}^{n+1} \times - \nabla \delta n_{\theta 1}^{n+1} + \frac{m_e}{m_i} q_i \nabla \cdot \mathbf{j}_{\theta 1}^{n+1},
\]
(23)
where \( n_{\theta 0}, \mathbf{B}_0 \) stands for the equilibrium ion density and magnetic field respectively. The bulk current distribution is assumed to be in diffusive equilibrium \( \mathbf{E}_0 = \mathbf{j}_0 \). The off-diagonal terms of ion stress tensor are neglected here, hence the ion stress tensor \( \Pi \), becomes the isotropic ion pressure \( P \). If \( \theta = 0 \), the method is a first-order accurate explicit scheme (forward Euler), while if \( \theta = 1 \), it is first-order accurate implicit (backward Euler), and when \( \theta = 0.5 \), it becomes the Crank–Nicolson method [31].

The previous first-order scheme by Chen and Parker [19] solves the fields in Fourier space and hence can not handle inhomogeneous equilibrium fields because of Fourier convolutions. This difficulty is overcome in the present second-order scheme by solving the fields in real space along the inhomogeneous direction for each Fourier mode in the uniform direction. The generalized Ohm’s law is separated into an equilibrium part and a perturbed part. This adds additional complexity into the coding, but it simplifies the implicit field solver [32]. The equilibrium part can be written as a large linear equation and solved by direct matrix inversion. Note that the matrix is static for each Fourier mode, therefore the matrix only needs to be LU-decomposed once at the preconditioning stage. This significantly reduces the computation time. The perturbed part is solved iteratively.

First we advance the particle position and velocity using the known field quantities \( \mathbf{E}^n \) and \( \mathbf{B}^n \).
\[
\begin{align*}
\mathbf{x}^* &= \mathbf{x}^n + \Delta t (1 - \theta) \mathbf{v}^n, \\
\mathbf{v}^* &= \mathbf{v}^n + \Delta t (1 - \theta) \mathbf{a}^n.
\end{align*}
\]
(24)
And the particle weight is
\[
\mathbf{w}^* = \mathbf{w}^n - \Delta t (1 - \theta) \left( \mathbf{v} \cdot \nabla + \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}} \ln f_0(\mathbf{x}^n, \mathbf{v}) \right).
\]
(25)
Usually the marker particle distribution is loaded as Maxwellian, therefore we can approximate the ion current as
\[
\delta j_{\theta 1}^{n+1} = q_i \int \delta j_1^{n+1} \mathbf{v}_j^{n+1} d\mathbf{v}_j = \delta j_1^* + q_i \Delta t \sum_j \frac{\delta j_1^*}{\mathbf{E}_j^{n+1}} (\mathbf{x}_j^{n+1}) \cdot \mathbf{v}_j^{n+1} \mathbf{v}_j^{n+1} \simeq \delta j_1^* + 0 \Delta t \frac{q_i^2}{m_i} \mathbf{E}_j^{n+1} \equiv J_1^*.
\]
(26)
For accuracy, we iterate on the differences between \( \delta j_1^{n+1} \) and \( J_1^* \). Typically, in linear simulations, 5 iterations are enough. Fig. 1 shows the needed iteration times to make the field solver converge to the desired order of accuracy for various values of \( \delta n_{\theta 1}/n_{\theta 0} \).

Once the \( \mathbf{E}_j^{n+1} \) is calculated from the generalized Ohm’s law, one can go back to Eqs. (18)–(20) and (22) to advance the particle information and update the magnetic fields.

4. Benchmarks

This section presents simulations of the shear and compressional Alfvén waves, the whistler waves, and the ion acoustic wave with Landau damping. Unless explicitly stated otherwise, all simulations assume an ion-to-electron mass ratio of \( m_i/m_e = 1837 \). Boundary conditions are periodic in all three dimensions. Unless explicitly mentioned otherwise, the grids is \( \times 32 \times 32 \) and the simulation domain is \( l_x \times l_y \times l_z = 628 \rho_l \times 628 \rho_l \times 1000 \rho_l \). The total number of particles used is 131072. The timestep is set as \( \Omega \Delta t = 0.05 \).
4.1. Shear and compressional Alfvén waves

The dispersion relation of the shear Alfvén wave for a cold plasma yields

\[ \omega^2 = \frac{1}{\omega_0} k^2 v_A^2, \tag{27} \]

which reduces to \( \omega^2 = (1 - \omega/\omega_0) k^2 / \beta_e \) in our scaled quantities. For results presented here, there is a uniform background magnetic field \( B_0 \) along \( z \)-direction. Fig. 2 shows the simulation results for a mode with \( k_\perp = 0, k \rho_i = 0.00628 \), which agrees with the dispersion relation very well.

The dispersion relation for the compressional Alfvén wave is \( \omega = k V_A \) or \( \omega = k / \sqrt{\beta_e} \) in dimensionless quantities. If \( \Omega \Delta t \) is small enough, this fast wave can be accurately captured. As shown in Fig. 3, the simulation for \( k_\parallel = 0, k \rho_i = 0.01 \) agrees with the theory very well.

4.2. Ion acoustic waves and Landau damping

The simulation of ion acoustic wave is compared with the kinetic dispersion relation. The ion acoustic wave has the following dispersion relation [23]

\[ D(k, \omega) = 1 + \sum \frac{e^2}{\epsilon_0 k m} \left( P \int_{-\infty}^{+\infty} \frac{\partial f_0 / \partial v}{\omega - kv} dv - \frac{\pi i}{k} \frac{\partial f_0}{\partial v} \right) = 0, \tag{28} \]

where the summation is over species. The electrostatic oscillation is propagating along a magnetic field in \( z \)-direction. According to the dispersion relation, ion acoustic wave is damped due to the resonance of particles with speeds close to the phase speed \( \omega / k \). This kinetic effect is known as Landau damping. The Landau damping from electrons is relatively small compared to that from ions by a factor of \( \sqrt{m_e / m_i} \), therefore only the ion Landau damping is considered here. The dispersion

Fig. 1. The iteration times needed for field solver to converge to an accuracy order of \( \Delta E_x / E_x = 10^{-9} \) (triangles) and \( 10^{-7} \) (diamonds).

Fig. 2. Comparison of simulated shear Alfvén wave frequency \( \omega \) vs \( \beta_e \) (diamonds) and the dispersion relation (solid line).
The relation is solved numerically for the case of $T_i = T_e = 0.2$ and the result is shown as the solid line in Fig. 4. The simulation is in good agreement with the kinetic theory.

Linearly, the initially excited waves will damp exponentially. Nonlinearly, the nonlinear energy interchange between waves and the trapped particles eventually stops this damping process [33]. As the phase-mixing of resonant particles becomes more complete, the less energy interchange occurs. The wave amplitude eventually saturates at a nonzero value.
which is directly related to the initial perturbation. Fig. 5(a) compares the linear and nonlinear runs with the same parameters. In the nonlinear regime, the wave amplitude reaches a constant residual value higher than the recurrence value in the linear run \([34,35]\). Fig. 5(b) shows the saturated electric field amplitude as a function of the initial perturbation amplitude. The dash line corresponds to the analytical solution of Ref. [36]. Although the analysis in Ref. [36] is based on the nonlinear Landau damping of the electron plasma wave, the governing equations are the same as in the present case. Therefore, the results apply in the present case without much modification. In our units, the saturation amplitude of the wave is determined by

\[
E_{\text{fr}} = \frac{4\pi^2 \nu_{\text{ph}}^2}{(\nu_{\text{ph}}^2 - \nu_{\text{th}}^2)} (E_{\text{in}} - E_{\text{fr}}) \approx 0.31 (E_{\text{in}} - 0.51)
\]

(29)
given \(\nu_{\text{ph}}^2 = 0.2, \nu_{\text{th}} = 1.35, k = 0.1\) in the simulation. Here \(E_{\text{fr}}\) is the critical value of the initial perturbation amplitude which distinguishes the linear and nonlinear regimes. The simulation agrees well with the theory. Note that in order to reduce the wave amplitude at which the recurrence phenomenon occurs, many more particles (2097152) are used in this simulation.

4.3. Numerical damping of the whistler waves

By dropping ion dynamics, the model is simplified and it becomes possible to investigate the numerical damping associated with the implicit time advance. To work out a numerical dispersion relation, we drop the ion current and the electron inertia in the generalized Ohm’s law. The reduced generalized Ohm’s law yields

\[
E = \frac{1}{\rho_e} (\nabla \times \delta B) \times B_0.
\]

(30)
Numerically, the above equation reads,

\[
E^{n+1} = \frac{1}{\rho_e} (\nabla \times \delta B^{n+1}) \times B_0.
\]

(31)
And the numerical Faraday’s law is

\[
\frac{\delta B^{n+1} - \delta B^n}{\Delta t} = -[\theta \nabla \times E^{n+1} + (1 - \theta) \nabla \times E^n].
\]

(32)
Assume the uniform background \(B_0\) is along \(z\)-direction, we consider the perturbed fields \(E^n, \delta B^n\) of the form

\[
E^n = E_e e^{i(kz - \omega t)},
\]

\[
\delta B^n = B_e e^{i(kz - \omega t)},
\]

where \(E_e\) and \(B_e\) are uniform. A Von Neumann stability analysis provides the corresponding numerical dispersion relation for whistler wave

\[
\begin{align*}
\tan(\omega t \Delta t) &= \frac{\beta_e^2}{1 - \beta_e^2 \rho_e (1 - \theta)} \\
\gamma \Delta t &= -\frac{1}{2} \ln \left( \frac{\left(1 - \beta_e^2 \rho_e (1 - \theta)\right)^2 + \beta_e^4 \rho_e^2}{1 + \beta_e^4 \rho_e^2 (1 - \theta) + \beta_e^2 \rho_e (1 - \theta)} \right)
\end{align*}
\]

(33)
By neglecting terms with an order of \((\Delta t)^2\) or higher \((\Delta t \ll 1)\), the above equations can be simplified as

\[
\begin{align*}
\omega_t &= \frac{k^2}{\rho_e}, \\
\gamma &= (\frac{1}{2} - \theta) (\frac{k^2}{\rho_e})^2 \Delta t.
\end{align*}
\]

We can see that the centering parameter \(\theta\) does not affect the real frequency (up to \((\Delta t)^2\)). But the imaginary part of the frequency \(\gamma\) depends on \(\theta\) directly. The implicit time advance damps the whistler wave when \(\theta > 0.5\) and drives it unstable when \(\theta < 0.5\). When \(\theta = 0.5\), the time-centered scheme introduces negligible numerical damping. Fig. 6 shows the numerical damping under different centering parameters for \(k_c = 0, k_i = 0.0628, \rho_e = 0.004\). The results agree with the numerical dispersion relation Eq. (33) very well. And as expected, different centering parameter produces the same real frequency, which is not shown here. Also omitted is the unstable regime of \(\theta < 0.5\). The grids used here is \(16 \times 16 \times 32\) and the total number of particles is \(131072\). The simulation domain is \(l_x \times l_y \times l_z = 628 \rho_i \times 628 \rho_i \times 100 \rho_i\).

5. Tearing mode simulation

The tearing mode is an important instability that occurs in plasmas with a sheared magnetic field, where small perturbations of the field lines would lead the plasma to a lower magnetic energy state via field bending and form magnetic islands.
There are various mechanisms to break the local magnetic flux conservation resulting in a magnetic topology change, which makes tearing mode a natural cause of magnetic reconnection [37]. Analysis of the tearing mode is extensive including resistive MHD, two-fluid and kinetic theories [22,38–40]. The simplest analysis of the tearing mode is done in the presence of finite resistivity, where the instability develops on a resistive time scale and is the so called resistive tearing mode. The linear theory of resistive tearing mode instability is well established by Furth, Killeen and Rosenbluth, the FKR theory [22].

The simulation starts with an initial configuration of Harris sheet equilibrium [41], the equilibrium magnetic field is

\[ B_0 = e_B + e_B \tan h(x/a), \quad (34) \]

where \( B_0 \) represents the uniform guide field, \( B_{0o} \) is the asymptotic Harris magnetic field, and \( a \) stands for the width of the current sheet. The coordinates are chosen as x in the direction of the gradients, y along the sheared Harris magnetic field, and z along the equilibrium current. The equilibrium distribution function is

\[ f_{eq} = n_{eq} \text{sech}^2 \left( \frac{x}{a} \right) \left( \frac{2\pi T_i}{m_i} \right)^{3/2} \exp \left[ -\frac{m(v_i^2 + q_e^2 + (v_i - v_e)^2)}{2T_i} \right] + n_b \left( \frac{2\pi T_s}{m_s} \right)^{3/2} \exp \left( -\frac{m^2 v^2}{2T_s} \right), \quad (35) \]

where \( v_{de} = 2T_s/qsB_{0o}a \) is the drift velocity, and \( n_b = n_{eq} \text{sech}^2 (x/a) \) and \( n_b \) stands for the Harris particle density and uniform background density respectively. The temperature of the Harris and background particles are chosen to be the same. Since the equilibrium current density is carried by the ion and electron drift velocity, the Ampere’s law \( \nabla \times B_0 = e(n_e(v_{de} - v_{de})) \) leads straightforwardly to the pressure balance \( B_{0o}^2/2\mu_0 = n_{eq}(T_i + T_e) \). The marker ion distribution is loaded as \( g_0 = (2\pi T_{io}/m_i)^{-3/2} \exp(-mv^2/2T_{io}) \), and according to Eq. (15), the weight equation reads

\[ \frac{dw_i}{dt} = \frac{q_i}{T_i} \left( (E - \frac{\eta}{\rho_e} \nabla \times B) \cdot \frac{\delta y}{g_0} f_{h} + n_b \right) - v_i \cdot \left( E + v_i \times \delta B - \frac{\eta}{\rho_e} \nabla \times B \right) f_{h} g_0. \quad (36) \]

The boundary conditions are periodic in \( y \) and \( z \) directions and a perfect conducting impenetrable wall in \( x \). Particles are reflected when they hit the wall. \( E = B_x = 0 \) at the wall. The results presented here use a mass ratio of \( m_i/m_e = 400 \). The total number of particles is 8388608.

We now investigate the dependence of the linear growth rate on the wave number \( k \) in \( y \) direction. The tearing mode develops a discontinuity in the derivative of magnetic flux in the vicinity of the neutral layer, which is usually characterized by the tearing instability parameter \( \Delta' = (d\psi_1/dx|_{0o} - d\psi_1/dx|_{o})/\psi_1(0) \), where \( \psi_1 \) is the perturbed flux eigenfunction outside the inner region. In most cases, \( \Delta' \) can only be calculated numerically. Fortunately, for Harris sheet equilibrium, there is an analytic value of \( \Delta' = \frac{2}{3} (\frac{\kappa}{\kappa_0} - ka) \). Moreover, if the finite \( \kappa_0 \) (length between the walls) is considered, the expression for \( \Delta' \) becomes [42]

\[ \Delta' = \frac{2}{3} (\frac{1}{\kappa_0} - ka) \frac{ka - \tanh(\kappa/2a) \tanh(k\kappa/2)}{ka \tanh(k\kappa/2) - \tanh(\kappa/2a)}. \quad (37) \]

In dimensionless quantities, the resistive-MHD linear growth rate can be written as as [22,23]

\[ \gamma = 0.55 \left( \frac{1}{\rho_e} \right)^{1/5} \Delta'^{4/5} \eta^{3/5} (kB_{0o}^2)^{2/5}. \quad (38) \]

Therefore, the Harris sheet is unstable with respect to tearing when \( \Delta' > 0 \), while it remains stable when \( \Delta' < 0 \).

Fig. 7 shows the full evolution of island width with various values of \( \Delta' \). Note the instability threshold is \( \Delta' > 0 \), which is confirmed by the simulation. We can clearly identify the linear growing, nonlinear growing and saturation stages. The island shrinking after saturation indicates the free energy stored in the initial configuration has been exhausted. Fig. 8 shows snap-
shots of the mode structure and the corresponding magnetic island at two different times. The discontinuity at the middle of the current sheet gradually disappears as the instability develops (shown by the eigenmode structure), which stands as a signature of the tearing mode saturation.

In Fig. 9, we show the linear growth rates as a function of $ka$. The simulation is in agreement with theory. When resistivity gets smaller than 0.001, the electron inertia effects becomes important and the results deviate from the simple resistive MHD theory.

The spatial convergence properties of the eigenfunction (perturbed $A_z$, the flux function) is shown in Fig. 10. The two grids produce similar results.

After a linear growth period, the tearing mode enters into the nonlinear regime. From Fig. 7, we can identify the Rutherford regime (around $\Omega_0 t \sim 150 \sim 200$) followed by the saturation (after $t \sim 230$). The Rutherford regime is characterized by an algebraic growth rate of the island width [28]:

$$dw/dt = 1.22\eta(\Delta' - \Delta'w).$$

which reduces to the conventional Rutherford equation when island width $w$ is small [24]. For Harris equilibrium, $\Delta' = 0.82$. As shown in Fig. 11, the simulated island growth rate scales linearly with $\Delta'$. The simulation agrees well with the theory.
There are numerous theoretical studies of the nonlinear saturation of the tearing mode [25–29,43,44]. Among these results, one simple interpretation of saturation is $\frac{dw}{dt} = 0$. According to Eq. (39), the saturation island width is

$$w_s = 1.22\Delta'$$  \hspace{1cm} (40)

In Fig. 12, the saturation island width is plotted for various values of $\Delta'$ calculated from Eq. (37). The saturation island width from the simulation (diamonds) is significantly smaller than the prediction (Eq. (40)) if we use the theoretical $\Delta'$ (Eq. (37)).
However, if we use the measured $D_0$ from the eigenmode structure plots, the predicted saturation island width (stars) agrees much better with the simulation. The reason is because the theoretical $D_0$ expression comes from asymptotic approximation of small $D_0$. When we measured the $D_0$ from the eigenmode structure at the moment of saturation, it is more consistent with the simulation thus leads to better agreement.

For larger values of $\Delta'$ ($\gtrsim 9$), secondary islands start to form and later coalesce with the main island [29]. The corresponding study should be addressed in another paper.

6. Summary

In this paper, we have presented a hybrid kinetic MHD model suitable for studying MHD scale plasmas using low order ion moments. This model has full kinetic ions and fluid electrons. Using Lorentz ions is partly motivated by the invalidation of gyrokinetic turbulence simulations at both long and short wavelengths [20,45]. Based on this model, a second-order accurate semi-implicit algorithm has been developed and implemented in slab geometry. The semi-implicit scheme eliminates the fast compressional waves and is numerically stable. The model advances the particle ions with fully electromagnetic equations, and fields are calculated through the generalized Ohm’s law and Faraday’s law. The electrons are assumed isothermal, and the quasi-neutrality provides the closure. To benchmark the code, we have simulated the linear shear and compressional Alfvén waves for a uniform plasma and the results agree with the theory. The ion acoustic wave and ion Landau damping are also correctly captured by the model. We have also reported the nonlinear Landau damping of the ion acoustic wave. In order to compare the second-order and first-order schemes, we work out the numerical dispersion relation for the whistler waves and compared it with the simulation. The simulation demonstrates that the time-centered second-order scheme introduces negligible numerical damping, which is in good agreement with the numerical analysis presented. Furthermore, the full evolution of the resistive tearing mode is investigated. The linear growth rate agrees with the resistive MHD prediction. Important nonlinear tearing dynamics including the Rutherford regime and the saturation are shown to be consistent with previous MHD studies.

This work represents a first step in developing a simulation suitable for MHD scale physics with full kinetic ions. To address the kinetic electron physics, more sophisticated electron models, e.g., drift-kinetic [19] and gyro-kinetic electrons, should be incorporated. These are topics of ongoing and future work.

References