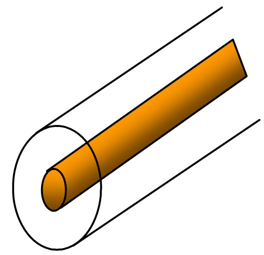
Equilibria of a cylindrical plasma

Consider a infinitely long cylinder of plasma with a strong axial magnetic field B_z (a great fusion device). Plasma pressure will cause the axial field inside to be smaller than the field outside as a result of the diamagnetic drift current. If this plasma, in addition, carries an axial current J_z, the **J** x **B** force will "squeeze" the plasma and the field inside can be larger than the field outside (paramagnetism). In this exercise we will examine equilibria of cylindrical plasmas. The first example is a plasma with pressure, but no axial current. The second example is a plasma with axial current but no pressure. The third example contains both axial current and pressure.



Cylindrical plasma column

I. Plasma cylinder with pressure and no axial current

In equilibrium, the J x B force must cancel the grad P_{plasma} force. The vector equation for equilibrium is:

$$\vec{J} \times \vec{B} - \vec{\nabla} P_{plasma} = 0$$

We will assume that the plasma is cylindrically symmetric (d/dθ = 0 and d/dz = 0). The applied magnetic field is B_z and the gradient in P_{plasma} is in the radial direction. The radial component of the vector equation above is

$$J_\theta B_z - J_z B_\theta - \frac{\partial P_{plasma}}{\partial r} = 0$$

We assume also that there is no axial current J_z.

The applied field B_z is assumed to be 1 Tesla. $B_{z0} := 1$ Tesla

The subscript zero for variables is used here to indicate a boundary condition, except for μ_0.

Define: $\mu_0 := 4 \cdot \pi \cdot 10^{-7}$

The external magnetic pressure is then: $P_{mag0} := \frac{B_{z0}^2}{2 \cdot \mu_0}$ $P_{mag0} = 3.979 \times 10^5$ N/m²

Assume a plasma radius of 1 m: $a := 1$ meters

Let the plasma pressure be 0.3 of this and vary parabolically with radius: $P_{plas}(r) := 0.3 \cdot P_{mag0} \cdot \left[1 - \left(\frac{r}{a}\right)^2\right]$ N/m²

If there is no axial current J_z, then the equilibrium equation gives a simple result for J_θ:

$$J_\theta = \frac{1}{B_z} \frac{\partial P_{plas}}{\partial r}$$

Ampere's law is: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$ and the θ component is: $-\frac{\partial B_z}{\partial r}=\mu_0 J_\theta$

After substituting for the diamagnetic current, Ampere's law becomes: $\frac{\partial B_z}{\partial r}=-\frac{\mu_0}{B_z}\frac{\partial P_{plas}}{\partial r}$

The problem is thus reduced to a single differential equation with a solution that can be found analytically. We will, however, find the solution by integrating the differential equation, in order to see how the problem can be solved computationally.

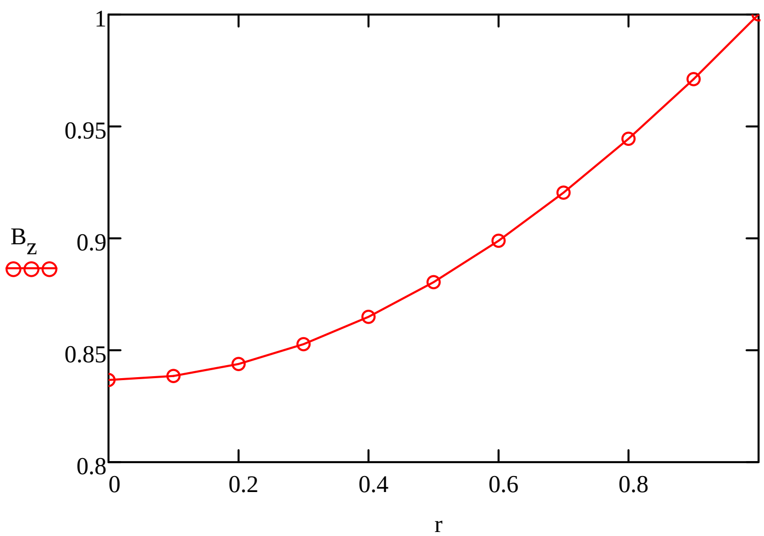
The Runge-Kutta method requires that we define the derivative of B_z as a function $D(r,B_z)$: $D(r,B_z) := \frac{-\mu_0}{B_z}\cdot\left(\frac{d}{dr}P_{plas}(r)\right)$

The applied magnetic field B_{z0} appears at the maximum radius r = a and is the boundary condition. The field $B_z(r)$ is then found by integrating from r = a to zero. This is done using the Runge-Kutta routine in Mathcad and the answer matrix will be M_{ans}.

$M_{ans} := \mathrm{rkfixed}(B_{z0}, a, 0, 10, D)$ We integrate from a to 0 in 10 steps.

Recall that the first columns of M_{ans} will have the values of r and the second column will have the values of B_z. $r := M_{ans}^{\langle 0\rangle}$ $B_z := M_{ans}^{\langle 1\rangle}$

The plot shows the diamagnetic reduction in B_z

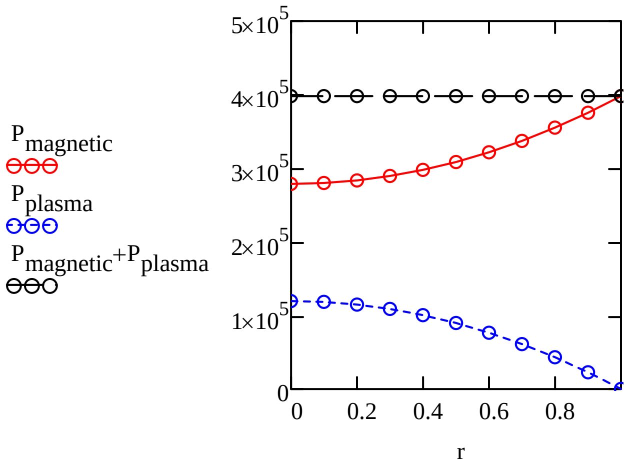


For plotting, we will need the plasma pressure as a function of radius: The new variable P_{plasma} is defined as a vector with the same number of elements as the radius r.

$$P_{plasma} := P_{plas}(r)$$

We similarly define a vector with the magnetic pressure:

$$P_{magnetic} := \frac{B_z^{\,2}}{2\cdot\mu_0}$$



$P_{magnetic} + P_{plasma} =$

	0
0	3.979·105
1	3.979·105
2	3.979·105
3	3.979·105
4	3.979·105
5	3.979·105
6	3.979·105
7	3.979·105
8	3.979·105
9	3.979·105
10	3.979·105

Try it: Compare the pressures above to the pressure of 1 standard atmosphere.

II. Plasma cylinder with axial current and no pressure gradient

In the absence of a pressure gradient, the equilibrium equation becomes:

$$\mu_0 \vec{J}\times\vec{B} = (\vec{\nabla}\times\vec{B})\times\vec{B} = 0$$

This equation is satisfied if **J** is parallel to **B**. The equation can be rewritten:

$$\vec{\nabla}\times\vec{B} = \lambda(r)\vec{B}$$

where λ is a coefficient of proportionality that can be a function of radius.

J is found from:

$$\vec{J} = \frac{\lambda(r)}{\mu_0}\vec{B} = 0$$

The θ component of the equation is:

$$-\frac{\partial B_z}{\partial r} = \lambda(r)B_\theta$$

The z component is:

$$\frac{1}{r}\frac{\partial}{\partial r}rB_\theta = \lambda(r)B_z$$

These two equations can be combined:

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}B_\theta = \lambda(r)\frac{\partial}{\partial r}B_z = -\lambda(r)^2 B_\theta$$

If λ(r) is simply a constant λ_0, then a solution for B_z is the zero order Bessel function $J_0(\lambda r)$and a solution for B_θ is the first order Bessel function $J_1(\lambda r)$.

Define a new variable rBtheta, written as one word, which is defined as the product of r and B_θ. The equilibrium equations are:

$$\frac{\partial B_z}{\partial r} = -\lambda(r)\frac{(rBtheta)}{r} \qquad \frac{\partial}{\partial r}(rBtheta) = \lambda(r) rB_z$$

Boundary conditions will be specified on the axis rather than at the plasma surface. The integration will begin at r = 0 rather than r = a. The starting values will be:

$rBtheta_0 := 0 \qquad Bz_0 := 1$

The parameter λ will be a constant: $\lambda_0 := 3.0$

The variable B is a vector with the two components of B: $B := \begin{pmatrix} rBtheta_0 \\ Bz_0 \end{pmatrix}$

The equilibrium equations in the form used by the Runge-Kutta routine are:

$$D(r, B) := \begin{pmatrix} \lambda_0 \cdot r \cdot B_1 \\ -\lambda_0 \cdot \frac{B_0}{r} \end{pmatrix} \qquad \begin{matrix} \frac{d}{dr} rB_\theta \\ \frac{d}{dr} B_z \end{matrix}$$

In order to avoid division by zero, we will begin the integration at r = 0.001 rather than zero:
The integration goes from r = 0.001 to r = 1 with 10 steps.

$$M := \text{rkfixed}\left[\begin{pmatrix} rBtheta_0 \\ Bz_0 \end{pmatrix}, 0.001, 1, 10, D\right]$$

The columns of the answer matrix will be put into variable with names that we recognize:

M =

	0	1	2
0	1·10-3	0	1
1	0.101	0.015	0.977
2	0.201	0.058	0.911
3	0.301	0.122	0.807
4	0.401	0.2	0.67
5	0.501	0.279	0.511
6	0.6	0.349	0.339
7	0.7	0.398	0.166
8	0.8	0.416	2.385·10-3
9	0.9	0.398	-0.142
10	1	0.339	-0.26

$r := M^{\langle 0 \rangle}$

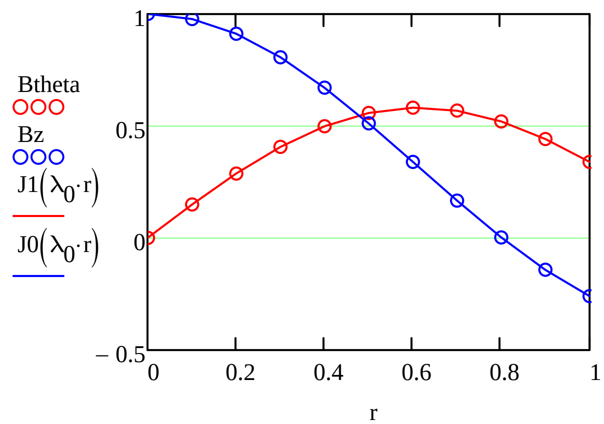
$rBtheta := M^{\langle 1 \rangle}$

$Bz := M^{\langle 2 \rangle}$

$Btheta := \overrightarrow{\frac{rBtheta}{r}}$ A vectorized equation.

A force-free equilibrium

The equilibrium that we have found is called a force-free equilibrium because there are no forces. Our Runge-Kutta solution for constant λ agrees with the Bessel function analytic solutions. The plotted profile applies to the reversed-field pinch fusion device which has an axial field that is reversed at the edge of the plasma.



The solution above is not realistic because the plasma is coldest and most resistive at the wall. Thus the current density should go to zero at the wall. If we construct a function λ(r) that goes to zero at the wall, then the current density at the wall is automatically zero.
Let's find a more realistic equilibrium with J = 0 at the wall using the new definitions:

$$\lambda(r) := \lambda_0 \cdot \left[1 - \left(\frac{r}{a}\right)^2\right]$$

$$D(r, B) := \begin{pmatrix} \lambda(r)\cdot r\cdot B_1 \\ -\lambda(r)\cdot\frac{B_0}{r} \end{pmatrix}$$

In order to avoid division by zero, we will begin the integration at r = 0.001 rather than zero: The integration goes from r = 0.001 to r = 1 with 10 steps.

$$M2_{ans} := \mathrm{rkfixed}\left[\begin{pmatrix} rBtheta_0 \\ Bz_0 \end{pmatrix}, 0.001, 1, 10, D\right]$$

The columns of the answer matrix will be put into variable with names that we recognize:

$$r := M2_{ans}^{\langle 0 \rangle} \qquad Bz := M2_{ans}^{\langle 2 \rangle}$$

$$rB\theta := M2_{ans}^{\langle 1 \rangle} \qquad Btheta := \overrightarrow{\frac{rB\theta}{r}}$$

And we will define the currents:

$$Jtheta := \frac{1}{\mu_0}\cdot\overrightarrow{(\lambda(r)\cdot Bz)}$$

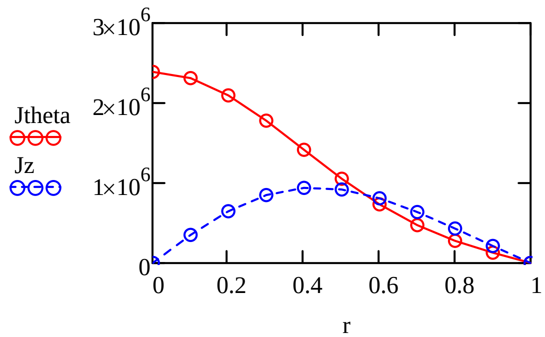
$$Jz := \frac{1}{\mu_0}\cdot\overrightarrow{(\lambda(r)\cdot Btheta)}$$

r rBθ Bz

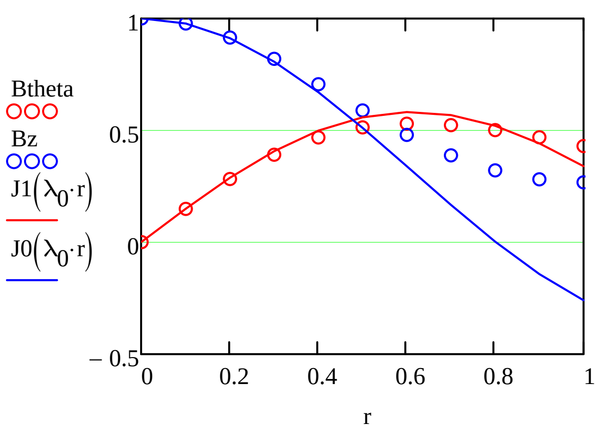
$M2_{ans} =$

	0	1	2
0	1·10-3	0	1
1	0.101	0.015	0.977
2	0.201	0.057	0.914
3	0.301	0.117	0.819
4	0.401	0.187	0.706
5	0.501	0.257	0.588
6	0.6	0.318	0.479
7	0.7	0.366	0.388
8	0.8	0.401	0.321
9	0.9	0.422	0.281
10	1	0.429	0.267

Plot of the components of the current J



Plot of the equilibrium magnetic profile using a parabolic λ(r) profile



The Bessel function solutions (solid lines) are plotted for comparison. These apply if λ is a constant.

Try it: Note that the axial magnetic field is no longer reversed. Increase the λ_0 value so that the reversed field pinch configuration (B_z reversed at the wall) is created.

Finding the current I_z from the current density J_z:

We have found the current density J_z at a set of radial locations. The axial current can be found by integrating J_z over the area, but for this we need a continuous function not a tabulated function. The tabulated function can be made continuous using spline interpolation:

Create the vector of coefficients vs: $vs := \text{cspline}(r, Jz)$

Define the function: $J(R) := \text{interp}(vs, r, Jz, R)$

Do the integration: $2\cdot\pi\cdot\int_0^a J(R)\cdot R\,dR = 1.718\times10^6$ amps

Our plasma carries about 2 million amps.

III. Plasma cylinder with current and a pressure gradient

A voltage applied at the ends of a long plasma column will drive a current. The conductivity of the plasma is greatest parallel to the field lines thus we can assumed our applied current is parallel to **B**. The diamagnetic current that is required for equilibrium is perpendicular to B. We will add the applied and diamagnetic current densities to obtain the total current density, and then integrate Ampere's law to find the magnetic configuration.

A. Diamagnetic currents J_d

From the equilibrium equation, we obtain:

$$\vec{J}_d = \frac{\vec{B} \times \vec{\nabla} P}{B^2}$$

The pressure gradient is only in the radial direction, so we obtain:

$$J_{d\theta} = \frac{B_z}{B^2}\frac{\partial P_{plas}}{\partial r} \qquad J_{dz} = -\frac{B_\theta}{B^2}\frac{\partial P_{plas}}{\partial r}$$

B. Applied currents

We must make an assumption regarding the distribution of the applied current. We will assume that the amplitude of the current density varies parabolically. In order not to disturb the force balance, we will assume that the applied current flows parallel to the magnetic field. The applied current density is then:

Define a central current density: $J_0 := 1.0 \cdot 10^6$ amps/m²

$$J_{applied}(r) := J_0 \cdot \left[1 - \left(\frac{r}{a}\right)^2\right] \qquad J_{applied\theta} = \frac{B_\theta}{B} J_0(r) \qquad J_{appliedz} = \frac{B_z}{B} J_0(r)$$

Write the components of Ampere's law:

$$\frac{1}{r}\frac{\partial}{\partial r} rB_\theta = \mu_0[J_{dz} + J_{appliedz}] = \mu_0\left[-\frac{B_\theta}{B^2}\frac{\partial P_{plas}}{\partial r} + \frac{B_z}{B} J_{applied}(r)\right]$$

$$\frac{\partial B_z}{\partial r} = -\mu_0[J_{d\theta} + J_{applied\theta}] = -\mu_0\left[\frac{B_z}{B^2}\frac{\partial P_{plas}}{\partial r} + \frac{B_\theta}{B} J_{applied}(r)\right]$$

Again us the variable rBtheta to represent r multiplied by B_θ.

Again define a vector B with rBtheta and Bz as the zeroth and first compontents:

$$B := \begin{pmatrix} rBtheta_0 \\ Bz_0 \end{pmatrix}$$

C. Integration of Ampere's law

We will find the equilibrium magnetic field profiles by integrating Ampere's law using the diamagnetic currents J_d that satisfy the equilibrium relation and the applied currents that create no additional force.

$$D3(r,B) := \begin{bmatrix} \mu_0 \cdot r \cdot \left[\dfrac{-\dfrac{B_0}{r} \cdot \left(\dfrac{d}{dr} P_{plas}(r)\right)}{\left(\dfrac{B_0}{r}\right)^2 + \left(B_1\right)^2} + \dfrac{B_1 \cdot J_{applied}(r)}{\sqrt{\left(\dfrac{B_0}{r}\right)^2 + \left(B_1\right)^2}} \right] \\ -\mu_0 \cdot \left[\dfrac{B_1 \cdot \left(\dfrac{d}{dr} P_{plas}(r)\right)}{\left(\dfrac{B_0}{r}\right)^2 + \left(B_1\right)^2} + \dfrac{\dfrac{B_0}{r} \cdot J_{applied}(r)}{\sqrt{\left(\dfrac{B_0}{r}\right)^2 + \left(B_1\right)^2}} \right] \end{bmatrix}$$

Note that B_0/r in these expressions is B_θ.

The above definition is the same as:

$$\frac{\partial}{\partial r} rB_\theta = \mu_0 r\left[-\frac{B_\theta}{B^2}\frac{\partial P_{plas}}{\partial r} + \frac{B_z}{B} J_{applied}(r)\right]$$

$$\frac{\partial B_z}{\partial r} = -\mu_0\left[\frac{B_z}{B^2}\frac{\partial P_{plas}}{\partial r} + \frac{B_\theta}{B} J_{applied}(r)\right]$$

Find the fields by integrating Ampere's law from 0.001 to 1 in 10 steps:

$$M3_{ans} := \text{rkfixed}\left[\begin{pmatrix} rBtheta_0 \\ Bz_0 \end{pmatrix}, 0.001, 1, 10, D3\right]$$

$M3_{ans} =$

	0	1	2
0	1·10-3	0	1
1	0.101	6.36·10-3	0.998
2	0.201	0.025	0.991
3	0.301	0.054	0.98
4	0.401	0.093	0.968
5	0.501	0.138	0.955
6	0.6	0.186	0.945
7	0.7	0.235	0.941
8	0.8	0.279	0.943
9	0.9	0.314	0.955
10	1	0.333	0.977

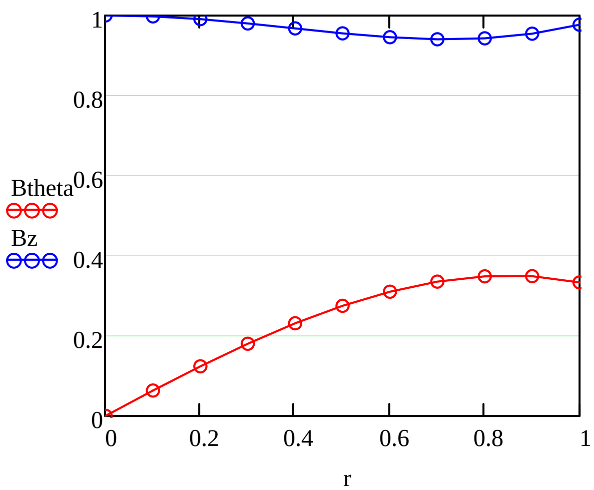
Use familiar variable names:

$r := M3_{ans}^{\langle 0 \rangle}$ $\quad Bz := M3_{ans}^{\langle 2 \rangle}$

$rB\theta := M3_{ans}^{\langle 1 \rangle}$ $\quad Btheta := \overrightarrow{\dfrac{rB\theta}{r}}$

$$Babs := \sqrt{Btheta^2 + Bz^2}$$

Plot of the equilibrium profile with a current of 0.5 MA



This profile with $B_\theta(a) << B_z$ is near to the profile for a tokamak plasma if it were converted to a straight cylinder.

Note that there is an inflection point in the Bz profile for this choice of P and $J_{applied}$.

Try it: Experiment with different values of current density and plasma pressure.

D. Test the equilibrium for accuracy

If there is an equilbrium, the J X B and Grad P forces are equal and opposite.
We can check that by plotting them individually and by plotting their sum.
The equilibrium equation is:

$$\left(J_{applied\theta} + J_{d\theta}\right)B_z - \left(J_{appliedz} + J_{dz}\right)B_\theta - \frac{\partial P_{plasma}}{\partial r} = 0$$

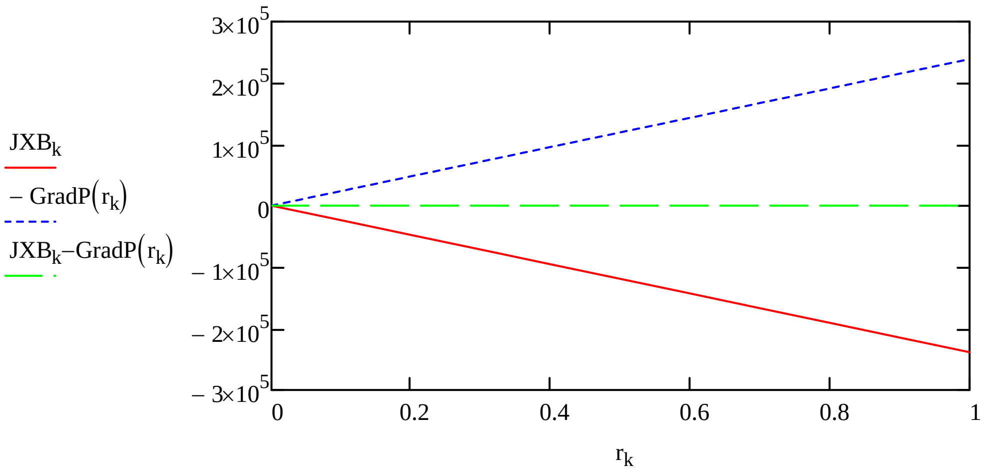
The gradient in pressure that we will need is: $\mathrm{GradP}(R) := \frac{d}{dR}P_{plas}(R)$

$k := 0 .. \mathrm{rows}(r) - 1$ Index for the number of radial locations

The J X B force at each ofthe k radial locations is:

$$\mathrm{JXB}_k := \left[\frac{\mathrm{Btheta}_k}{\mathrm{Babs}_k}\cdot J_{applied}(r_k) \cdots + \frac{\mathrm{Bz}_k}{\left(\mathrm{Babs}_k\right)^2}\cdot \mathrm{GradP}(r_k)\right]\cdot \mathrm{Bz}_k - \left[\frac{\mathrm{Bz}_k}{\mathrm{Babs}_k}\cdot J_{applied}(r_k) \cdots + \frac{-\mathrm{Btheta}_k}{\left(\mathrm{Babs}_k\right)^2}\cdot \mathrm{GradP}(r_k)\right]\cdot \mathrm{Btheta}_k$$

Plot showing that the J X B force and Grad P force are equal and opposite:



The near-zero sum at right shows that the method for finding the equilibrium is accurate.

$JXB_k - GradP(r_k) =$

0
$-7.276 \cdot 10^{-12}$
$-7.276 \cdot 10^{-12}$
$1.455 \cdot 10^{-11}$
$-1.455 \cdot 10^{-11}$
$-1.455 \cdot 10^{-11}$
0
$-2.91 \cdot 10^{-11}$
$2.91 \cdot 10^{-11}$
0
$-8.731 \cdot 10^{-11}$

Reference:

Paul M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, Cambridge, 2006), p. 408.