The harmonic oscillator: motion in a potential well

Consider a one dimensional electrostatic potential well:
\[ \phi(x) := \alpha \cdot x^2 \]
The electric field is:
\[ E(x) := -2 \cdot \alpha \cdot x \]

The equations of motion for an ion are the two first order equations:
\[ \frac{dv(t)}{dt} := \frac{q}{m} \cdot E(x) \]
\[ \frac{dx(t)}{dt} := v(t) \]

Define \( Z \) containing \( x \) in the first position and \( v \) in the second position. \( Z \) will initially contain the starting values of \( x \) and \( v \).

\[
Z := \begin{pmatrix}
0 \\
1
\end{pmatrix}
\]

Keeping in mind that \( Z_0 \) is \( x \) and \( Z_1 \) is \( v \).

We will let:
\[ q := 1 \quad m := 1 \]

Define \( \alpha \) and \( E \):
\[ \alpha := 0.5 \quad E(x) := -2 \cdot \alpha \cdot x \]

The derivatives of \( x \) and \( v \) are:
\[ DZ(t,Z) := \begin{pmatrix}
Z_1 \\
\frac{q}{m} \cdot E(Z_0)
\end{pmatrix} \]

Using the Runge-Kutta routine \texttt{rkfixed} to integrate the equations. The frequency in radians/second of the oscillator is unity and we will follow it for 10 oscillations, thus the ending time is \( 20\pi \). We will divide this interval into 300 points. The answers go in a matrix \( M \).

\[
npoints := 300 
M := \text{rkfixed}(Z,0,20\pi,npoints,DZ)
\]

\( M \) will contain \( t \) in column zero, \( X \) (same as \( Z_0 \)) in column 1, and \( V \) (same as \( Z_2 \)) in column 2.

We will define new vectors \( T \), \( X \), and \( V \) as these columns of \( M \) and then make plots.
These plots show that we indeed have a harmonic oscillator:

Phase error:
The "spring constant" (the force/distance) for our oscillator is $2\alpha$, which is unity. Thus the period is $2\pi$ and our ending time of $20\pi$ should result $x$ being zero at the end. Instead the $x$ value is:

$$X_{\text{npoints}} = -9.916 \times 10^{-4}$$

Which is slightly less than zero. This is a small phase error.

Amplitude error:
Our harmonic oscillator has no damping, thus the sum of the potential energy ($0.5x^2$) and kinetic energies ($0.5mv^2$) should be a constant. Let’s plot this sum and see what we get:

$$j := 0 \ldots \text{npoints}$$

The relative change in energy is very small.

Try it: How are the phase and amplitude errors affected by doubling npoints (halving the time step)? Is the error second order in the time step or fourth order?