Numerical integration of derivatives by three methods

We will compare integration by (1) Euler's method, (2) the midpoint method, and (3) Runge-Kutta. The function $4x^3$ will be integrated. The integral is $x^4$, so we will compare the results of the integration with $x^4$. We will integrate from $x=0$ to $x=10$ in 10 steps with $\Delta x=1$.

1. **Euler's method (error proportional to $\Delta x$)**

The slope of $y(x)$ evaluated at $x$ is used to find the value of $y$ at $(x+\Delta x)$:

Euler's method uses:  
$$y_n := y_{n-1} + \Delta x \frac{dy}{dx}(x_n)$$

$h(x) := 4 \cdot x^3$  $h(x)$ is the derivative of $y$

Define the grid:  
$$\text{imax} := 10 \quad \Delta x := \frac{10}{\text{imax}} \quad i := 0 \ldots \text{imax} \quad x_i := i \cdot \Delta x$$

Initialize the answer vector $g$, then iterate

$$g := g_{\text{imax}} \leftarrow 0$$

for $i = 1 \ldots \text{imax}$

$$g_i := g_{i-1} + \Delta x \cdot h(x_{i-1})$$

$g_{\text{imax}} = 8100$  which is significantly less than 10,000.

The points found by Euler's method lie significantly below the curve $x^4$.

2. **Midpoint method: (error proportional to $(\Delta x)^2$)**

The midpoint method uses:  
$$y_n := y_{n-1} + \Delta x \frac{d}{dx}y(x_n + 0.5 \Delta x)$$

The derivative is evaluated half-way to the next point, and is "centered" on the interval of interest.

$$g := g_{\text{imax}} \leftarrow 0$$

for $i = 1 \ldots \text{imax}$

$$g_i := g_{i-1} + \Delta x \cdot h\left(x_{i-1} + \frac{\Delta x}{2}\right)$$

$g_{\text{imax}} = 9950$

The midpoint method gives a more accurate answer.
3. Runge Kutta method: error proportional to \((\Delta x)^4\)

The function `rkfixed` does the Runge Kutta integration starting at `xstart` and ending at `xend` using `npoints` as the number of intervals.

- `xstart := 0` starting x
- `xend := 10` ending x
- `npoints := 10` number of intervals, one less than the number of grid points.
- `ystart := 0` this is the starting value of `y` (the constant of integration)
- `dy(x, y) := 4\cdot x^3` defines the derivative. The `dy` used by `rkfixed` MUST have two arguments.

The function `rkfixed` does the integration and places the `x` and `y` values in the matrix `F`.

\[
F := \text{rkfixed}(ystart, xstart, xend, npoints, dy)
\]

<table>
<thead>
<tr>
<th>x</th>
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<tbody>
<tr>
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\[
F_{\text{imax}, i} = 10000 \quad \text{Runge Kutta has integrated this function without error.}
\]

Try it:

1. What happens in the first two methods if the number of points is doubled so that \(\Delta x = 0.5\)?

2. How far off is the integration by Euler's method if 100,000 points are used \((\text{imax}=10^5)\)? Can you demonstrate that the error is proportional to \(\Delta x\)?