Plotting magnetic field lines

What is the trajectory of a magnetic field line? A ball follows a trajectory \( x(t), y(t), z(t) \). A field line has no time dependence, it is "just there." For a plot, we need to replace the variable \( t \) with the distance along the line \( s \). Then the line can be expressed as \( x(s), y(s), \) and \( z(s) \). As we vary \( s \), these functions tell us where we will find the point on the field line corresponding to that value of \( s \).

At each point in space the B field points in a direction with components \( B_x, B_y \) and \( B_z \). So if we move along the B field line a small distance \( ds \), then the end of the B field line moves from \((x, y, z)\) to \((x + dx, y + dy, z + dz)\). The fraction of the distance \( ds \) that is in the \( x \) direction is \( B_x / |B| \). So if we move along the line a distance \( ds \), then \( x \) progresses by \((B_x/|B|) ds\). So now we see that the distance of \( x(s) \) from the starting point is the integral of \((B_x/|B|) ds\).

The expression for \( x(s) \) is

\[
x(s) = x_0 + \int_0^s \frac{B_x}{|B|} ds'
\]

where \( ds' \) is a dummy variable.

The B field of a wire

If a straight wire is in the \( z \) direction, then the B field is in the \( \theta \) direction. The \( x \) component of a unit vector in the \( \theta \) direction is \(-y/r\) and the \( y \) component is \( x/r\). So a unit vector in the magnetic field direction is \( \mathbf{U} = \left[ -\frac{B_\theta}{|B|}(y/r), \frac{B_\theta}{|B|}(x/r), 0 \right] \). Now we just integrate these with \( ds \). Keep in mind that \( B_\theta \) is a function of \( r \): \( B_\theta(r) = \mu_0 I / 2\pi r \). Let’s pretend that \( \mu_0 \) is one so that we don’t need scientific notation for our answers.

\( I := 5 \) is the current. We will let \( B_z \) be zero for now: \( B_z := 0 \).

The vector \( \mathbf{X} \) will contain the starting point for the line:

\[
\mathbf{X} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

\( X_0 \) is \( x \), \( X_1 \) is \( y \), \( X_2 \) is \( z \).

Our starting point will be

\[
\begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}
\]

In terms of the components of \( \mathbf{X} \), the radius is

\[
\rho(\mathbf{X}) := \sqrt{\left(X_0\right)^2 + \left(X_1\right)^2}
\]

The \( \theta \) field is

\[
B_\theta(\mathbf{X}) := \frac{\mu_0 I}{2\pi \rho(\mathbf{X})}
\]

The absolute value of B is

\[
B(\mathbf{X}) := \sqrt{B_\theta(\mathbf{X})^2 + B_z^2}
\]
The components of the unit vector along $B$

\[
U_x(X) := \frac{(-X)_1 B_\theta(X)}{r(x) - B(X)} \quad U_y(X) := \frac{X_0^2 B_\theta(X)}{r(x) - B(X)} \quad U_z(X) := \frac{B_z}{B(X)}
\]

Below put the three functions to be integrated into the 3-vector Derivs.

\[
\text{Derivs}(s, X) := \begin{pmatrix} U_x(X) \\ U_y(X) \\ U_z(X) \end{pmatrix}
\]

We will put lots of points in our lines. $\text{npoints} := 200$

The program "rkfixed" below uses the Runge-Kutta method to integrate the derivatives Derivs a distance of 6 units and divides this into npoints intervals. The $B$ field line is a circle, of course, and it does not quite close because 6 is not quite $2\pi$.

\[
\text{M0} := \text{rkfixed}(X, 0, 6, \text{npoints}, \text{Derivs}) \quad \text{The matrix M will contain s, x(s), y(s) and z(s)}
\]

We only need to plot columns 1, 2, and 3. Column zero is not needed.

You can grab the axes and tilt this 3-d plot.

**Try it:** Does the gap in the ring close if you make npoints bigger?
The B field of a straight wire PLUS a constant field

Now suppose we do this again and add some $B_z$. This will cause the B field line to "move" along the $z$ axis as it goes around in the $\theta$ direction. The result is a helix.

$$B_z := 1$$

I have to repeat the formulas above so that they are evaluated with the new $B_z$.

$$B(X) := \sqrt{B\theta(X)^2 + B_z^2}$$

$s := 0$ We will start at: $X := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The components of the unit vector are:

$$U_x(X) := \frac{(-X) \cdot B\theta(X)}{r(X) \cdot B(X)}$$

$$U_z(X) := \frac{B_z}{B(X)}$$

$$U_y(X) := \frac{X \cdot B\theta(X)}{r(X) \cdot B(X)}$$

Derivs1 is the 3-vector of derivatives. Integrate them with rkfixed.

$$M1 := \text{rkfixed}(X, 0, 40, \text{npoints}, \text{Derivs1})$$

It really is a helix:

You can grab the corners of the 3-d plot and rotate it. Then you can really tell it is a helix.

Try it: What happens if you change $B_z$ to 0.1?

Autoscale for the $z$ axis must remain turned off to see any difference.