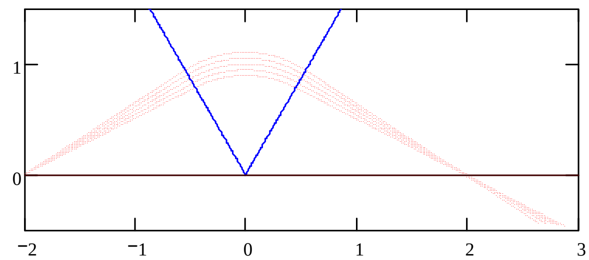
Deflection of a particle beam by a bending magnet

The Runge-Kutta method will be used on the Lorentz equations of motion to follow a charged particle beam through a B field between the poles of a bending magnet. The pole faces of the magnet are an equilateral triangle. Bending magnets have been used to separate beams of ions according to their mass in order to identify the isotopes of elements and their relative abundances. Mass spectrographs were designed to operate analogously to prism spectrographs for light. A good design causes all of the ions entering through a slit to exit through another slit if their masses are the same. Hence the magnet must focus the beam as well as bend it.



Deflection of a particle beam (in red) by a bending magnet (outlined in blue).

The particle coordinates will be a 4-vector with subscripts 0 and 1 for x and y, and subscripts 2 and 3 for v_x and v_y.

The vector Zstart will hold the starting coordinates of the particle. In the example at right the particles starts at x = -4, y = 0 with velocities v_x = 1 and v_y = 0. The meanings of the 4 components are listed to the right of Zstart.

$$\text{Zstart} := \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{matrix} x \\ y \\ v_x \\ v_y \end{matrix}$$

The pole faces of the magnet will be an equilateral triangle. The half-angle between the faces is 30 degrees or $\Theta = \pi/6$. The B field will have a magnitude of 1.0 between the pole faces and will be zero outside the pole faces. The region with the field is $|x| < y \tan \Theta$.

The angle between one face of the triange and the vertical axis: $\Theta := \dfrac{\pi}{6}$

The magnetic field as a function of position:
Recall that Z_0 is x and Z_1 is y.
If Z is within the magnet, B = 1, otherwise B = 0.

$$B(Z) := \text{if}\left[\left(\left|Z_0\right| < Z_1 \cdot \tan(\Theta)\right), 1.0, 0\right]$$

For example: $B\left(\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = 0$ $\quad B\left(\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}\right) = 1$

The field B is independent of Z_2 and Z_3.

The point (2,0) is outside the magnet and the point (0,2) is within the magnet.

The Lorentz equations of motion in two dimensions:

We will let the charge q = 1 and the mass m =1 to avoid scientific notation: $q := 1$ $m := 1$

DZ is the derivative of the 4-vector Z.
The bottom 2 terms, the accelerations, are the Lorentz force divided by the mass. This definition of the derivatives will be used by the Runge Kutta integrator.

$$DZ(t,Z) := \begin{pmatrix} Z_2 \\ Z_3 \\ \frac{q}{m} \cdot Z_3 \cdot B(Z) \\ -\frac{q}{m} \cdot Z_2 \cdot B(Z) \end{pmatrix} \begin{matrix} dx/dt \\ dy/dt \\ dv_x/dt \\ dv_y/dt \end{matrix}$$

Find and plot the particle trajectory

We will choose a trajectory so that the particle enters the magnet perpendicular to the face. This means that the initial velocity vector has an angle of Θ. The magnitude of the velocity will be 1.0.

$$Zstart := \begin{pmatrix} -2 \\ 0 \\ 1.0 \cdot \cos(\Theta) \\ 1.0 \cdot \sin(\Theta) \end{pmatrix} \begin{matrix} x \\ y \\ v_x \\ v_y \end{matrix}$$

For accuracy, we will want the trajectory to be divided so finely that there will be many points within the magnet. If we seek accuracy (in bending angle) of order 10^{-2}, then there should be about 100 points. The width of the magnet is about 1 unit and the velocity is about 1 unit, so we will use a time step of 0.01 which will place the points in the trajectory about 0.01 unit apart. These points will be so close together that a point plot will appear to be a continuous line.

Let t be the total time interval: $t := 5.4$

This value was determined by trial and error to give a nice plot.

This definition for the number of integration points gives the desired accuracy: $npoints := \frac{t}{0.01}$

Integrate the equation of motion with Runge Kutta:

The list of points that is generated by the Runge Kutta integrator is put in an answer matrix M.

$$M := rkfixed(Zstart, 0, t, npoints, DZ)$$

The answer matrix M is labelled here with the meaning of the columns.

M =

	t	x	y	v_x	v_y
	0	1	2	3	4
0	0	-2	0	0.866	0.5
1	0.01	-1.991	5·10-3	0.866	0.5
2	0.02	-1.983	0.01	0.866	0.5
3	0.03	-1.974	0.015	0.866	0.5
4	0.04	-1.965	0.02	0.866	0.5
5	0.05	-1.957	0.025	0.866	0.5
6	0.06	-1.948	0.03	0.866	0.5
7	0.07	-1.939	0.035	0.866	0.5
8	0.08	-1.931	0.04	0.866	0.5
9	0.09	-1.922	0.045	0.866	0.5
10	0.1	-1.913	0.05	0.866	...

Plot of one trajectory:

For plotting, it will be helpful to define the left and right boundaries of the bending magnet x1 and x2:

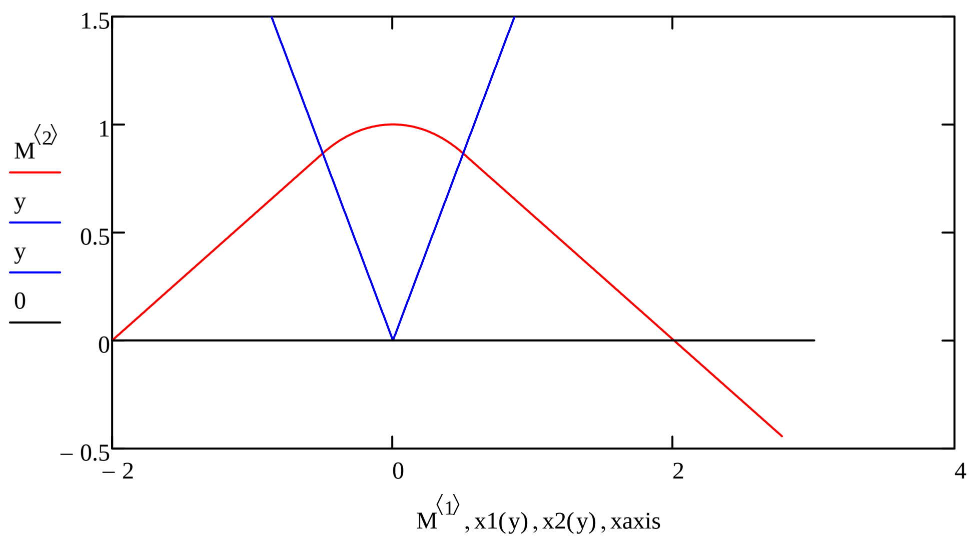
$y := 0, 0.01 .. 1.5$

$x1(y) := y \cdot \tan(\Theta)$ $\qquad$ $x2(y) := -y \cdot \tan(\Theta)$

It is also useful to have the horizontal axis defined:

$xaxis := -2, -1.9 .. 3$

The plot below shows that the trajectory is bent within the magnet and is straight outside of the magnet.



Trajectories for a divergent beam

An entrance slit is useful for limiting the spread in the starting positions of the beam particles. A beam has a divergence, which is a spread in the angles of the velocity vectors of the particles. If the magnet is made focusing, then particles from the entrance slit with slightly different angles of incidence will all pass through the exit slit. If the magnet is not focusing the number of particles passing through the exit slit will be smaller and thus harder to detect. Another problem is that an isotope of a different mass can pass through the exit slit if it enters with a velocity vector different of that of the isotope with the desired mass. We will show that the magnet is focusing by finding trajectories for a range of angles of incidence.

Define 5 angles of incidence θ_k that are near to the Θ used above:

$$k := 0 .. 4 \qquad \theta_k := \Theta \cdot \left(1 + \frac{k-2}{20}\right)$$

θ_k in degrees:

$$\theta = \begin{pmatrix} 27 \\ 28.5 \\ 30 \\ 31.5 \\ 33 \end{pmatrix} \cdot \text{deg}$$

Define $Zstart2_k$ as the 5 starting vectors:

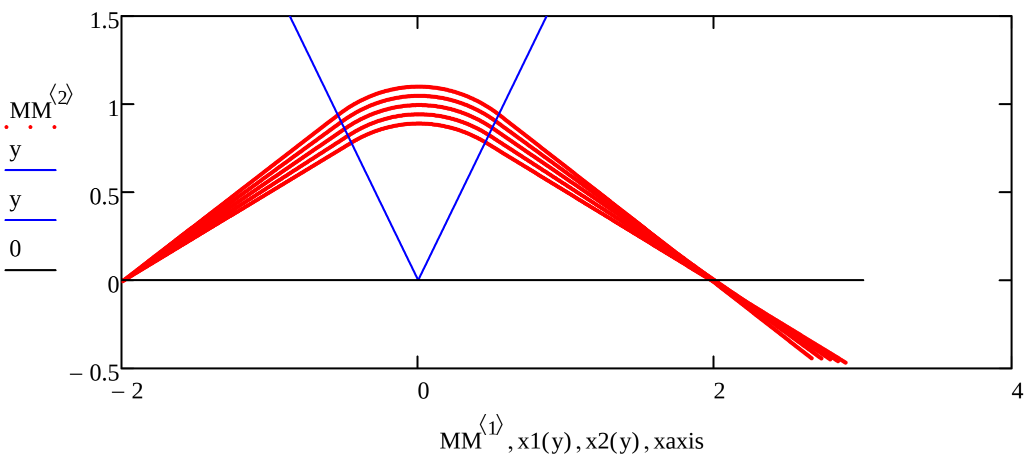
$$Zstart2_k := \begin{pmatrix} -2 \\ 0 \\ 1 \cdot \cos(\theta_k) \\ 1 \cdot \sin(\theta_k) \end{pmatrix}$$

The matrices $M2_k$ will have the 5 individual trajectories and the matrix MM will have them all.

The lines below run the Runge Kutta integrator 5 times for the 5 trajectories, then combine the trajectories into one matrix MM for plotting:

$$M2_k := \text{rkfixed}(Zstart2_k, 0, t, \text{npoints}, DZ) \qquad MM := \text{stack}(M2_0, M2_1, M2_2, M2_3, M2_4)$$

This plot of the 5 trajectories shows that they converge at a focus that is on the horizontal axis. Analysis of bending and focusing magnets shows that this is a property of magnets with faces 60 degrees apart if B is adjusted so that the Larmor radius originates at the apex of the triangle.



Try it: Investigate a magnet with 90 degrees between the faces by changing Θ to $\pi/4$ and changing the B field to 0.707.

Try it: Find the spread in the trajectories at the focal point by finding where the trajectories cross the horizontal axis.

Try it: Suppose that the edge rays deviate from the central ray by a small angle $\Delta\theta$. The position of the ray at the exit slit is independent of $\Delta\theta$ to first order, but not to second order. Increase the angular spread in the rays and show that the size of the focal point increases.

Reference:

A. Septier, *Focusing of charged particles* (Academic Press, New York, 1967), in two volumes.