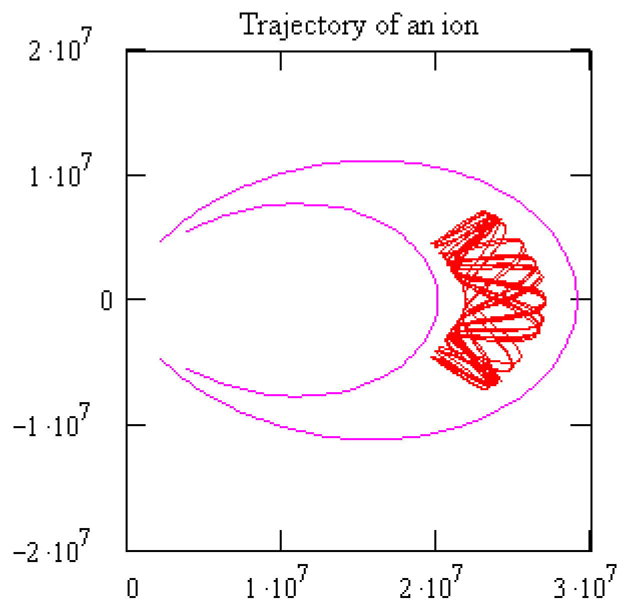
Ion motion in the earth's magnetic field: Van Allen's belts

We will plot the trajectory in 3-d of an oxygen ion in the earth's magnetic field. Oxygen ions are heavier than protons and will have a Larmor radius that is easier to see in the plots.

This exercise is in SI units.



$ORIGIN := 1$

Our vectors will have subscripts 1,2 and 3 for x,y, and z.

Begin at a starting point X that is 20,000 km from the earth along y.

$$X := \begin{pmatrix} 0 \\ 2\cdot 10^7 \\ 0 \end{pmatrix} \quad \begin{matrix} \text{x} \\ \text{y (in meters)} \\ \text{z} \end{matrix}$$

$$R(X) := \sqrt{(X_1)^2 + (X_2)^2 + (X_3)^2}$$ radius in spherical coordinates

Magnetic moment of the earth: $M := -7.84\cdot 10^{15}$ Tesla meters^3

$$B(X) := \begin{bmatrix} \dfrac{(3\cdot X_1\cdot X_3)\cdot M}{R(X)^5} \\ \dfrac{3\cdot X_2\cdot X_3\cdot M}{R(X)^5} \\ \left[\dfrac{3\cdot (X_3)^2}{R(X)^2} - 1\right]\cdot \dfrac{M}{R(X)^3} \end{bmatrix}$$

At left are the vector components of a dipole field from Jackson's "Classical Electrodynamics,"

Test this formula:

$$B(X) = \begin{pmatrix} 0 \\ 0 \\ 9.8\times 10^{-7} \end{pmatrix}$$

B is in the z direction around the equator and is about 0.01 Gauss or 10^{-6} T.

$$U(s,X) := \begin{pmatrix} \dfrac{B(X)_1}{|B(X)|} \\ \dfrac{B(X)_2}{|B(X)|} \\ \dfrac{B(X)_3}{|B(X)|} \end{pmatrix}$$

U is a unit vector along B. This vector is needed for plotting B.

Test this:

$$U(s,X) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

B is in the z direction above the equator.

To make a field line plot, we will just integrate the unit vector pointing along B and see where it goes.

$$X := \begin{pmatrix} 0 \\ 20\cdot 10^6 \\ 0 \end{pmatrix}$$

What field line passes though a point 20,000 km above the equator?
To find out, integrate the unit vector using the Runge Kutta routine, starting at the location X, given at left.

$$M2 := \text{rkfixed}\left(X, 0, 21\cdot 10^6, 40, U\right)$$

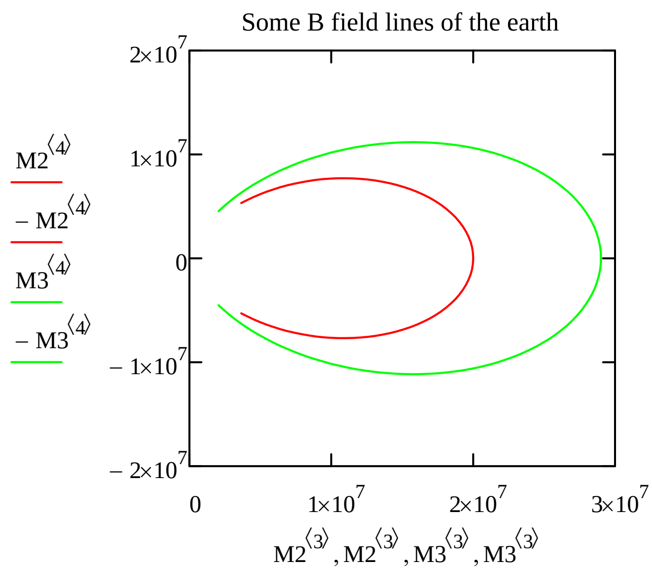
M2 is an answer matrix containing s, x, y, z. s is the distance along the field line, and x,y, z are the coordinates of points along the field line.

$$X := \begin{pmatrix} 0 \\ 29\cdot 10^6 \\ 0 \end{pmatrix}$$

This is another field line a little further from the earth

$$M3 := \text{rkfixed}\left(X, 0, 3.5\cdot 10^7, 60, U\right)$$

The lengths of the lines that have been plotted are 21 and 26 thousand km, which was found by trial and error. These lengths were chosen to end the field lines near the poles.



We know this plot is up-down symmetric. We can solve for the two field lines in the lower half plane then make a second plot upside down to get the upper half plane.

Let's stack these four lines into one array for plotting later

$$MM := \text{stack}\left(M2^{\langle 4\rangle}, -M2^{\langle 4\rangle}, M3^{\langle 4\rangle}, -M3^{\langle 4\rangle}, M2^{\langle 4\rangle}, -M2^{\langle 4\rangle}, M3^{\langle 4\rangle}, -M3^{\langle 4\rangle}\right)$$

$$NN := \text{stack}\left[M2^{\langle 3\rangle}, M2^{\langle 3\rangle}, M3^{\langle 3\rangle}, M3^{\langle 3\rangle}, (-M2)^{\langle 3\rangle}, (-M2)^{\langle 3\rangle}, (-M3)^{\langle 3\rangle}, (-M3)^{\langle 3\rangle}\right]$$

Set up the equations for the particle trajectory

Pick starting X and V vectors for the ion:

We will stack the derivatives of x, y, and z above the derivatives of V_x, V_y and V_z and call the new 6-vector DZ, below. The numbers at right were found by trial and error. They were selected to give nice looking plots.

$$X := \begin{pmatrix} 0 \\ 27\cdot 10^{6} \\ 0 \end{pmatrix} \quad V := \begin{pmatrix} 1\cdot 10^{7} \\ 0 \\ 3\cdot 10^{6} \end{pmatrix}$$

$Z := \text{stack}(X, V)$

$$\text{qmratio} := \frac{1.6\cdot 10^{-19}}{16\cdot 1.67\cdot 10^{-27}}$$

charge to mass ratio of an O ion

These two definitions let us recover X and V from the stack Z:

$X(Z) := \text{submatrix}(Z, 1, 3, 1, 1)$

$V(Z) := \text{submatrix}(Z, 4, 6, 1, 1)$

DZ is the derivative of the 6-vector Z:

$$DZ(t, Z) := \begin{bmatrix} Z_4 \\ Z_5 \\ Z_6 \\ \text{qmratio}\cdot(V(Z) \times B(X(Z)))_1 \\ \text{qmratio}\cdot(V(Z) \times B(X(Z)))_2 \\ \text{qmratio}\cdot(V(Z) \times B(X(Z)))_3 \end{bmatrix}$$

dx/dt
dy/dt
dz/dt

dVx/dt
dVy/dt
dVz/dt

The derivatives at left of Vx, Vy and Vz are from the Lorentz equations of motion.

What is the Larmor radius of the ion:

$$\text{LarmorRadius} := \frac{V(Z)_1}{\left(\text{qmratio}\cdot B(X(Z))_3\right)}$$

$\text{LarmorRadius} = 4.193 \times 10^{6}$

This is value is not representative because we selected an initial v too large, but the larger Larmor radius will show up nicely in the plots.

The gyro frequency is $\Omega := \text{qmratio}\cdot B(X(Z))_3$ $\Omega = 2.385$

The local Bz field is: $B(X(Z))_3 = 3.983 \times 10^{-7}$

Our time interval should be divided more finely than the gyration period.

Let t be the total time interval $t := 80$ seconds

The number of iterations needed is about 4 per gyro period or 8 π per orbit.

$\text{npoints} := \text{ceil}(4\cdot\Omega\cdot t)$ $\text{npoints} = 764$ (ceil is a rounding function to make integers)

Now integrate to find the ion trajectory::

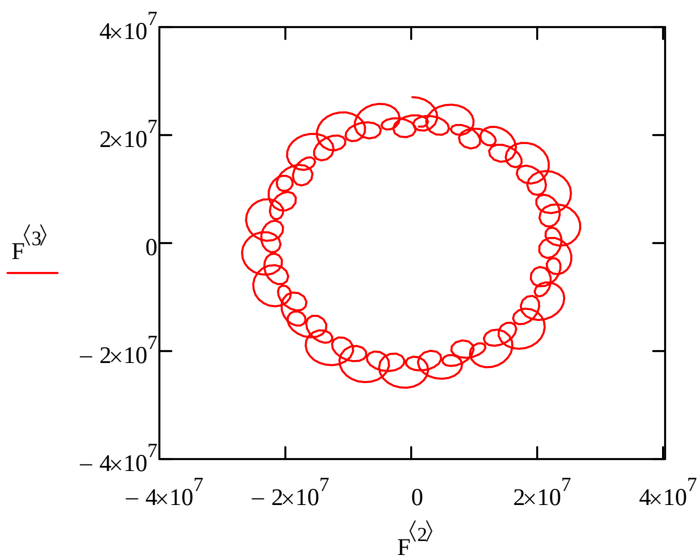
$F := \text{Rkadapt}(Z, 0, t, \text{npoints}, DZ)$

The reason for using *Rkadapt* rather than *Rkfixed* is discussed in the notes at the end.

F =

	1 (t)	2 (x)	3 (y)	4 (z)	5 (Vx)	6 (Vy)	7 (Vz)
1	0	0	$2.7\cdot10^{7}$	0	$1\cdot10^{7}$	0	$3\cdot10^{6}$
2	0.105	$1.037\cdot10^{6}$	$2.687\cdot10^{7}$	$3.126\cdot10^{5}$	$9.7\cdot10^{6}$	$-2.482\cdot10^{6}$	$2.957\cdot10^{6}$
3	0.209	$2.01\cdot10^{6}$	$2.648\cdot10^{7}$	$6.163\cdot10^{5}$	$8.79\cdot10^{6}$	$-4.871\cdot10^{6}$	$2.829\cdot10^{6}$
4	0.314	$2.855\cdot10^{6}$	$2.586\cdot10^{7}$	$9.026\cdot10^{5}$	$7.241\cdot10^{6}$	$-7.046\cdot10^{6}$	$2.629\cdot10^{6}$
5	0.419	$3.503\cdot10^{6}$	$2.502\cdot10^{7}$	$1.165\cdot10^{6}$	$5.017\cdot10^{6}$	$-8.841\cdot10^{6}$	$2.379\cdot10^{6}$
6	0.524	$3.882\cdot10^{6}$	$2.403\cdot10^{7}$	$1.401\cdot10^{6}$	$2.107\cdot10^{6}$	$-1\cdot10^{7}$	$2.133\cdot10^{6}$
7	0.628	$3.924\cdot10^{6}$	$2.296\cdot10^{7}$	$1.616\cdot10^{6}$	$-1.393\cdot10^{6}$	$-1.015\cdot10^{7}$	$1.997\cdot10^{6}$
8	0.733	$3.582\cdot10^{6}$	$2.195\cdot10^{7}$	$1.829\cdot10^{6}$	$-5.116\cdot10^{6}$	$-8.841\cdot10^{6}$	$2.157\cdot10^{6}$
9	0.838	$2.873\cdot10^{6}$	$2.117\cdot10^{7}$	$2.087\cdot10^{6}$	$-8.216\cdot10^{6}$	$-5.767\cdot10^{6}$	$2.869\cdot10^{6}$
10	0.942	$1.927\cdot10^{6}$	$2.079\cdot10^{7}$	$2.455\cdot10^{6}$	$-9.43\cdot10^{6}$	$-1.332\cdot10^{6}$	$4.277\cdot10^{6}$
11	1.047	$9.954\cdot10^{5}$	$2.089\cdot10^{7}$	$2.997\cdot10^{6}$	$-7.913\cdot10^{6}$	$3.068\cdot10^{6}$	$6.081\cdot10^{6}$
12	1.152	$3.498\cdot10^{5}$	$2.138\cdot10^{7}$	$3.716\cdot10^{6}$	$-4.154\cdot10^{6}$	$5.91\cdot10^{6}$	$7.538\cdot10^{6}$
13	1.257	$1.545\cdot10^{5}$	$2.206\cdot10^{7}$	$4.541\cdot10^{6}$	$4.463\cdot10^{5}$	$6.675\cdot10^{6}$	$8.015\cdot10^{6}$
14	1.361	$4.293\cdot10^{5}$	$2.272\cdot10^{7}$	$5.355\cdot10^{6}$	$4.67\cdot10^{6}$	$5.751\cdot10^{6}$	$7.357\cdot10^{6}$
15	1.466	$1.096\cdot10^{6}$	$2.322\cdot10^{7}$	$6.048\cdot10^{6}$	$7.861\cdot10^{6}$	$3.766\cdot10^{6}$	$5.747\cdot10^{6}$
16	1.571	$2.03\cdot10^{6}$	$2.349\cdot10^{7}$	$6.535\cdot10^{6}$	$9.764\cdot10^{6}$	$1.244\cdot10^{6}$	...

This view looks down from above the pole. It is the x,y projection:

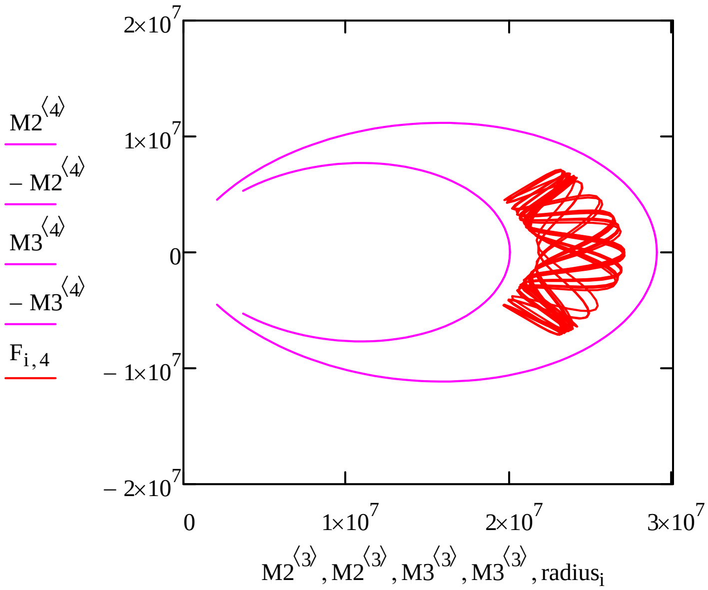


The ion drifted around the equator a little more than once.

Next we will make a plot of the motion projected onto the r,z plane.

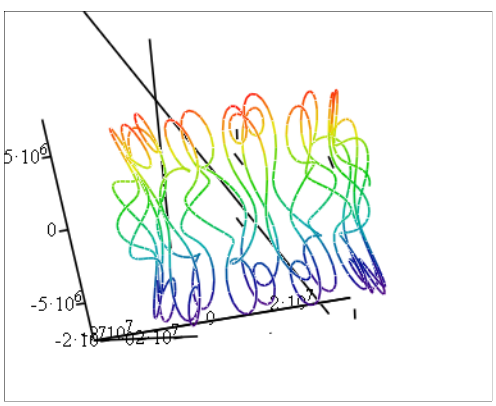
$i := 1, 2 .. npoints$

$radius_i := \sqrt{(F_{i,2})^2 + (F_{i,3})^2}$



This is a plot of the orbit projected onto the r,z plane

An interactive 3-d view, just grab a corner and pull:



$\left(F^{\langle 2\rangle}, F^{\langle 3\rangle}, F^{\langle 4\rangle}\right)$

Notes:

If the particle gets too close to the poles, the integration by Runge Kutta will fail. That is because the magnetic field is larger near the pole and so is the gyro frequency. Near the poles the time steps are not small enough to resolve the orbital motion. The integrator used above is the Rkadapt integrator which uses adaptive step size. This integrator automatically decreases the step size if the functions are changing too rapidly. This only partially fixes the problem.

If V_z is increased to values several times larger than 3 x 10^6, the program loses accuracy. Every bounce should reflect at the same distance from the poles. The accumulation of error causes the orbits to gradually change the distance at which they are mirrored.