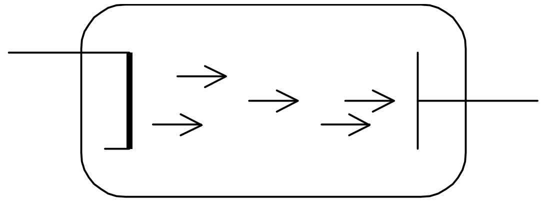
Child's law



In the diode vacuum tube in the figure, electrons are emitted from a heated cathode and flow to a positively biased anode.

Two observations have been made. 1) With the cathode heated a small amount, the current is equal to the current emitted from the cathode. It does not change with anode voltage because all of the emitted electrons are collected. 2) With large cathode heating, the current varies as the 3/2 power of voltage. **Why does the current vary with voltage?**

At high levels of cathode emission, the cloud of electrons in front of the cathode generates a negative potential causing some of the electrons to return to the cathode. The number going to the anode is determined by the space charge of the electrons in the gap between the anode and cathode. The number of electrons can be so great that Debye shielding reduces the electric field at the cathode nearly to zero. The diode can be modeled by using Poisson's equation in the region between the electrodes.

These are Poisson's equation changed to two first order equations so that Runge Kutta can be used.

$$\frac{d\phi}{dx} = -E$$

$$\varepsilon_0 \frac{dE}{dx} = nq = \frac{J}{v}$$

where $J = n\,q\,v$ is the current and we find the charge density from $nq = J / v$.

Define some familiar variables: $q := 1.6\cdot 10^{-19}$ $\varepsilon_0 := 8.854\cdot 10^{-12}$ $m := 9.11\cdot 10^{-31}$

The starting values for ϕ and E at the sheath boundary will be

$$y := \begin{pmatrix} 0 \\ .01 \end{pmatrix}$$

ϕ is y_0
E is y_1

We have made E = 0.01 because we are looking for the space charge limited solution in which E is nearly shielded out at the cathode. Later we can change E to see if this value affects the answer.

The starting value for J is $J := 1$ amps / m^2 This exercise is in SI units.

We assume that the cathode temperature is 2000 - 3000 K so that the mean electron energy is ~0.2 eV plus the additional energy from falling through the potential drop.

The electron velocity as a function of position is then $\sqrt{\frac{2\cdot q\cdot(y_0 + 0.2)}{m}}$ where y_0 is the potential $\phi(x)$.

The differential equations are:

$$DY(x,y) := \begin{bmatrix} -y_1 \\ \dfrac{-1}{\sqrt{\dfrac{2\cdot q\cdot(y_0+0.2)}{m}}}\cdot\dfrac{J}{\varepsilon_0} \end{bmatrix}$$

The meaning is:

$d\phi/dx = -E$

$dE/dx = nq = J/v$

Number of iterations:

$npoints := 200$

$i := 0, 1 .. npoints$

The anode-cathode separation is $d := 0.1$ meters

Start and end points $x1 := 0$ $x2 := d$

The Runge-Kutta $Y := rkfixed(y, x1, x2, npoints, DY)$

The final value for the potential is

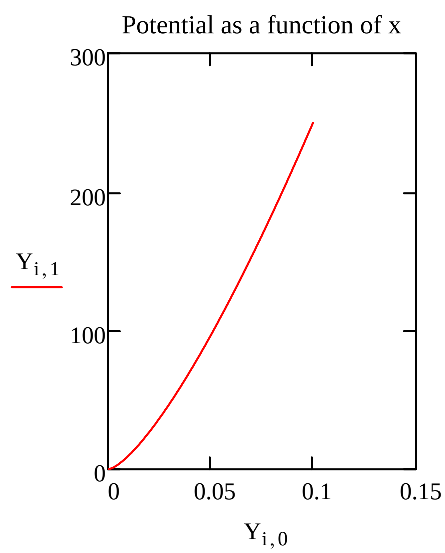
$Y_{npoints,1} = 249.796$ volts

It takes a voltage of about 250 V across a 0.1 m gap to create a current density of 1 A/m^2.

x, y_0 or ϕ, y_1 or E

Y =

	0	1	2
0	0	0	0.01
1	$5\cdot10^{-4}$	0.052	-204.793
2	0.001	0.199	-374.79
3	0.002	0.421	-509.79
4	0.002	0.704	-619.704
5	0.003	1.038	-712.219
6	0.003	1.414	-792.274
7	0.004	1.829	-863.044
8	0.004	2.276	-926.648
9	0.005	2.754	-984.553
10	0.005	3.26	-1037.818
11	0.006	3.792	-1087.226
12	0.006	4.347	-1133.376
13	0.007	4.924	-1176.734
14	0.007	5.523	-1217.669
15	0.008	6.142	...



Click the table and scroll down to see more.

Try it: What happens to the final voltage value for different values of J?

Method of solving a differential equation for different values of coefficient

Suppose we want to solve Poisson's equation for the voltage required for different values of J. The Runge Kutta routine in Mathcad does not allow us to change the J used inside of the derivatives DY without typing DY again. We cannot make DY a function of J.

There is a way to work around this!

We can add an additional term in the vector y and put the value of J in it. The derivative of J (now called y_2) is set equal to zero so that J does not change inside of the Runge Kutta routine. This makes our answers a function of J.

$$\text{ystart}(J) := \begin{pmatrix} 0 \\ 0.01 \\ J \end{pmatrix}$$

The current J is now the third component of the vector y. Our starting y can be made a function of J but we cannot make DY a function of J.

$$DY(x,y) := \begin{bmatrix} -y_1 \\ \dfrac{-1}{\sqrt{\dfrac{2\cdot q\cdot (y_0 + 0.2)}{m}}}\cdot\dfrac{y_2}{\varepsilon_0} \\ 0 \end{bmatrix}$$

The meaning is:

$d\phi/dx = -E$

$dE/dx = nq = J/v$ The variable y_2 is the current J.

$dJ/dx = 0$ Put zero here so that J does not change.

The Runge-Kutta $Y(J) := \text{rkfixed}(\text{ystart}(J), x1, x2, \text{npoints}, DY)$

For J = 1 we have the same answer as before: $Y(1)_{\text{npoints},1} = 249.796$

Below is an analytic solution to our problem that we can compare to the Runge Kutta answer.

$$i(\phi) := \frac{4}{9}\cdot\sqrt{\frac{2\cdot q}{m}}\cdot\left(\frac{\varepsilon_0}{d^2}\right)\cdot\phi^{\frac{3}{2}}$$

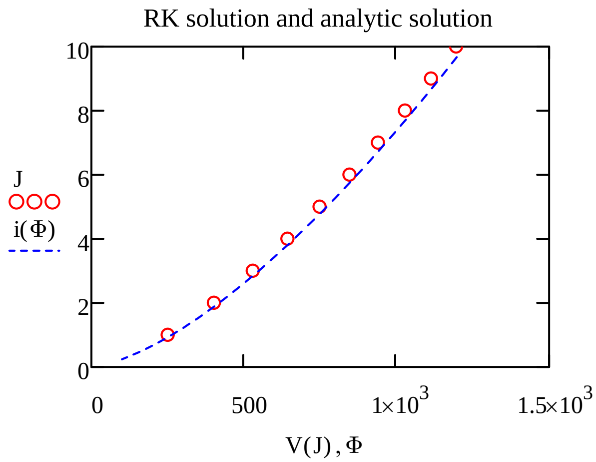
This is called both Child's law and the Langmuir-Child law. Child found the answer first but Langmuir popularized it.

Since Y is now a function of J, we can make a plot of the final voltage as a function of the current. For the theory, we make a plot of the current as a function of the voltage.

Create a range of J values for the plot of $\phi(J)$ $J := 1, 2 .. 10$

Create a range of Φ for the theory plot of $i(\phi)$ $\Phi := 100, 150 .. 1200$

Create a simpler variable name for the Runge Kutta answer $V(J) := Y(J)_{\text{npoints},1}$



The Runge Kutta solution and the analytic solution agree pretty well.

Try it:
Does the initial electron energy of 0.2 eV affect the answer? For example, is there a significant change if the 0.2 is changed to 0.5 or 0.1?
Does the initial value for E that we assumed, 0.01, have an effect? For example, is the final potential changed if we change the initial E to 0.1? 0.001?
Is the answer sensitive to the number of Runge Kutta steps, npoints?

The initial value of E and the initial energy do not appear in Child's law. This suggests that these values are not important. The analytic solution for Child's law is derived using E = 0 at the cathode.

Reference:
Chen's textbook, Chapter 8 through 8.2.4.