Landau's dispersion relation and Landau damping

The dispersion relation for electrostatic plasma waves is

$$\varepsilon(k,\omega) := 1 - \frac{\omega_p^2}{k^2}\cdot\int_{-\infty}^{\infty}\frac{df}{dv}\cdot\frac{1}{v-\frac{\omega}{k}}\,dv$$

where f (v) is the distribution function, ω_p is the plasma frequency, v_t is the thermal velocity, $v_t^2 = 2T/m$, and T is the temperature in energy units. The black square means that the expression is not evaluated.

The distribution function $f(v_x, v_y, v_z)$ is assumed to be Maxwellian. The distribution may be integrated over v_y and v_z to obtain

$$F(u) := \frac{1}{\sqrt{\pi v_t^2}}\cdot\exp\left(-u^2\right)$$

where $u = v_x/v_t$. The dispersion relation is then:

$$\varepsilon(k,\omega) := 1 - \frac{\omega_p^2}{k^2 v_t}\cdot\int_{-\infty}^{\infty}\frac{dF}{du}\cdot\frac{1}{u-\frac{\omega}{k\cdot v_t}}\,dv$$

or

$$\varepsilon(k,\omega) := 1 - \frac{\omega_p^2}{k^2 v_t^2\cdot\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}\frac{-2\cdot u\cdot e^{-u^2}}{u-\frac{\omega}{k\cdot v_t}}\,du$$

This can be rewritten as:

$$\varepsilon(k,\omega) := 1 - \frac{k_D^2}{k^2}\cdot W(\zeta)$$

$$\zeta := \frac{\omega}{k\cdot v_t} \qquad k_D := \frac{\sqrt{2}\cdot\omega_p}{v_t}$$

with the definitions

$$W(\zeta) := \frac{-1}{\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}\frac{x\cdot e^{-x^2}}{x-\zeta}\,dx$$

and, in terms of functions in Chen's textbook:

$$W(\zeta) := \frac{1}{2}\cdot\left(\frac{d}{d\zeta}Z(\zeta)\right)$$

The real part of the W function, Wr, is defined as the Principal Value integral:

$$Wr(\zeta) := \frac{-1}{\sqrt{\pi}}\cdot\left(\int_{-\infty}^{\zeta-\delta}\frac{x\cdot e^{-x^2}}{x-\zeta}\,dx + \int_{\zeta+\delta}^{\infty}\frac{x\cdot e^{-x^2}}{x-\zeta}\,dx\right)$$

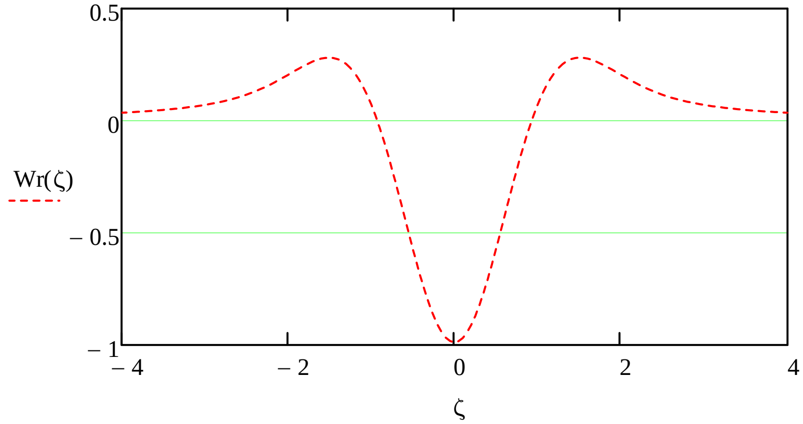
$$\delta := 0.01$$

where δ is a small quantity which is used to avoid the singularity.

The value of the integral is nearly independent of the value of δ along as δ is small.

$\zeta := -4, -3.95 .. 4$

Plot of the real part of W(ζ)



Remember that the dispersion relation is:

$$\varepsilon(k,\omega) := 1 - \frac{k_D{}^2}{k^2} \cdot W(\zeta)$$

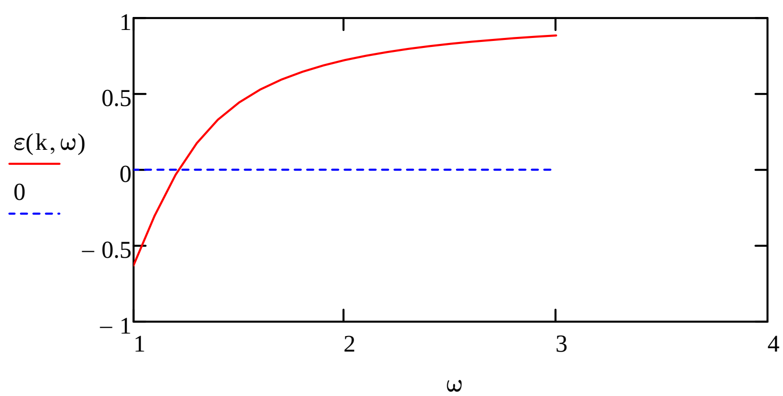
Let's simplify the math by letting: $\omega_p := 1$ $\quad v_t := 1$

The dispersion relation *ω*(k) is found from the roots of

$$\varepsilon(k,\omega) := 1 - \frac{2 \cdot \omega_p{}^2}{k^2 \cdot v_t{}^2} \cdot Wr\left(\frac{\omega}{k \cdot v_t}\right)$$

Let's find ε(k,ω) for $\quad k := 0.5$

Use these trial values for ω: $\quad \omega := 1, 1.1 .. 3$



ε(k,ω) crosses zero near 1.2, so this is the value of ω that satisfies the dispersion relation for k = 0.5.

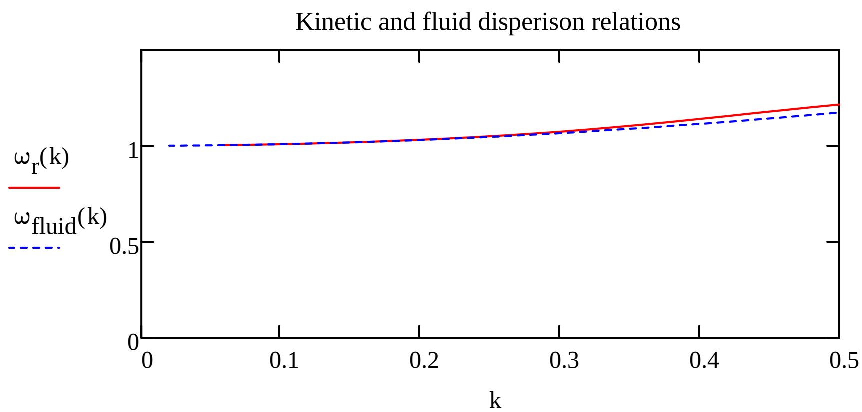
Now let's find ω for other values of k using the root finder. We will search between ω = 1 and ω = 2.

$\omega_r(k) := \text{root}(\varepsilon(k,\omega),\omega,1,2)$ $\omega_r(k) = 1.214$ This is near the 1.2 we found above.

Try it: Type in a different value for the δ that is used in the principal value integral and see if $\omega_r(k)$ is changed. Does it matter if δ is changed from 0.01 to 0.1 or 0.001?

Plot $\omega_r(k)$ for these values of k: $k := 0.02, 0.04 .. 0.5$

The wave frequency from fluid theory of waves is: $\omega_{fluid}(k) := \sqrt{\omega_p^2 + \frac{3}{2}\cdot k^2\cdot v_t^2}$



The fluid theory for electrostatic waves and the kinetic theory give nearly the same result for small k.

How does the principal value integral compare with the power series?

Here is the principal value integral again:

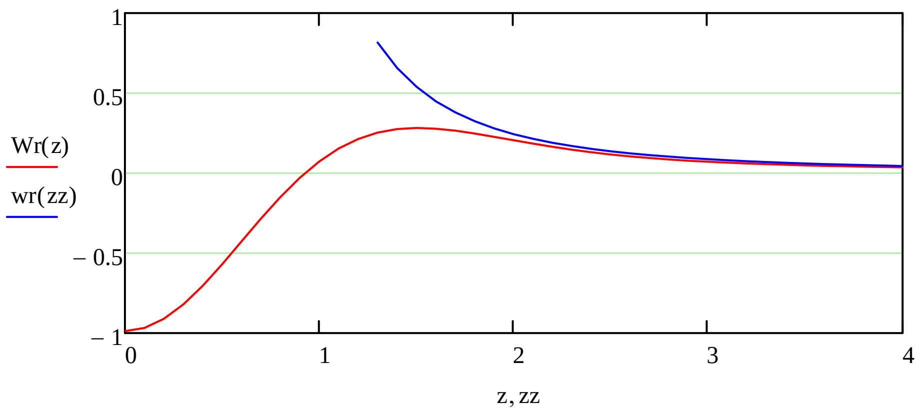
$$Wr(\zeta) := \frac{-1}{\sqrt{\pi}}\cdot\left(\int_{-\infty}^{\zeta-\delta}\frac{x\cdot e^{-x^2}}{x-\zeta}\,dx + \int_{\zeta+\delta}^{\infty}\frac{x\cdot e^{-x^2}}{x-\zeta}\,dx\right)$$

Here is the power series expansion of the denominator for small x/ζ.

$$wr(\zeta) := \frac{1}{\sqrt{\pi}}\cdot\int_0^{\infty} e^{-x^2}\cdot\left(1+\frac{2x}{\zeta}+\frac{3x^2}{\zeta^2}\right)\cdot\zeta^{-2}\,dx$$

In the figure on the next page, these two expressions are compared.

Define a range of values to be used for ζ: $z := 0, 0.1 .. 4$ $zz := 1.3, 1.4 .. 4$

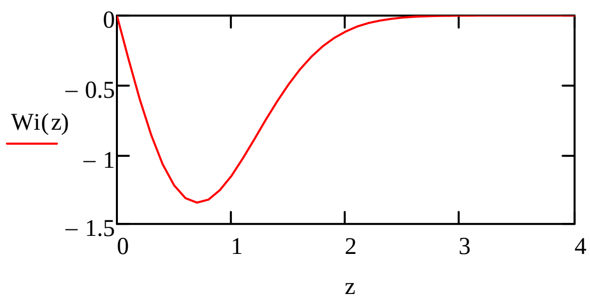


The two curves converge for large values of ζ for which the power series is valid.

The Landau damping is found from the imaginary part of W

Now let's look at the imaginary part of W(ζ) which we will call Wi(ζ).

$Wi(\zeta) := -\pi \cdot \zeta \cdot e^{-\zeta^2}$ where $\zeta^2 = \omega^2 / k^2 v_t^2$

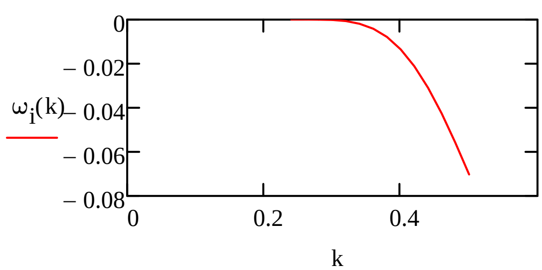


$$\lambda_d := \frac{v_t}{\sqrt{2} \cdot \omega_p}$$

Using Wi, it is easily shown that ω_i is:
(ω_i is the imaginary part of ω)

$$\omega_i(k) := -\sqrt{\frac{\pi}{8}} \cdot \frac{1}{(k \cdot \lambda_d)^3} \cdot \omega_r(k) \cdot e^{\frac{-1}{2 \cdot (k \cdot \lambda_d)^2} - \frac{3}{2}}$$

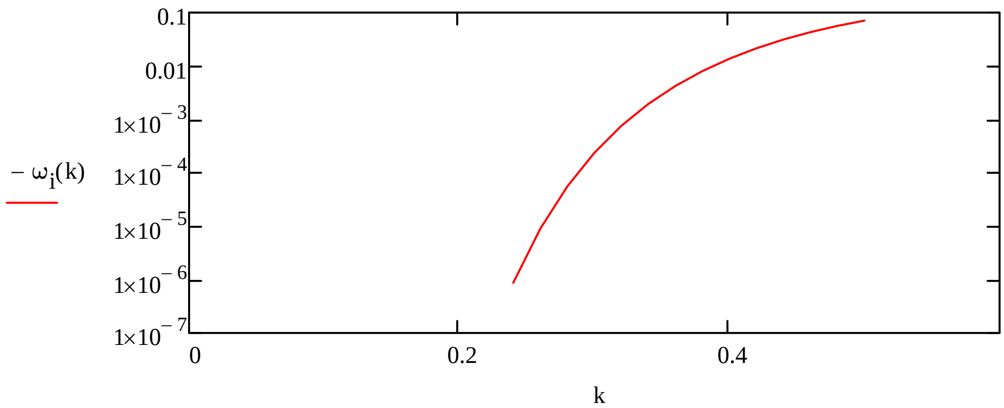
Plot ω_i(k) for these k values: $k := 0.24, 0.26 .. 0.5$



Plot of the imaginary part of ω as a function of the wavenumber k. First a linear plot then a logarithmic plot. Waves with larger k (short wavelength) are heavily damped.

Now plot again on a log scale (and change the sign before taking the logarithm).

Landau damping as a function of wavenumber:



Waves with k > 0.4 are strongly damped and those with k < 0.3 (long wavelength) have little damping.