Two stream instability

For an introduction to the instability, see Chapter 6.6 of the plasma physics textbook by Chen.

We imagine a plasma with cold electrons moving relative to cold ions.

The dispersion relation is $1 - F(k,\omega) := 0$

where $F(k,\omega) := \frac{\omega_{pe}^2}{(\omega - k\cdot v)^2} + \frac{\omega_{pi}^2}{\omega^2}$

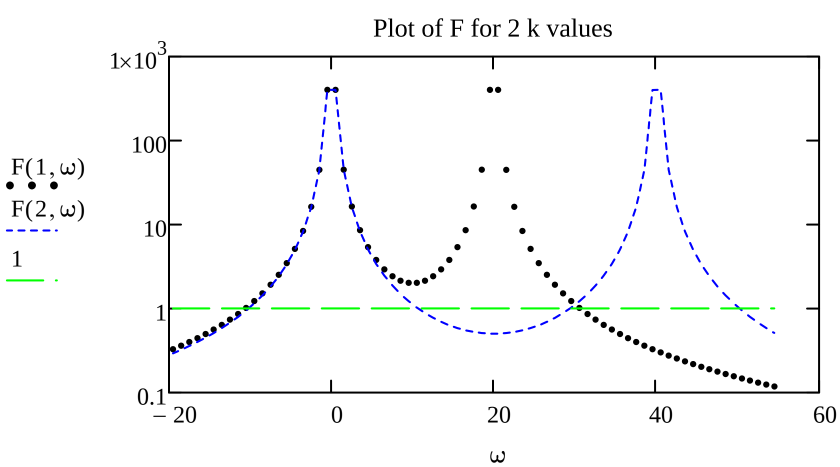
The little black boxes indicate that evaluation has been disabled.
Let's give the variables some values:

$\omega_{pe} := 10$ $\omega_{pi} := 10$ $k := 1$ $v := 20$

$$F(k,\omega) := \frac{\omega_{pe}^2}{(\omega - k\cdot v)^2} + \frac{\omega_{pi}^2}{\omega^2}$$

Now we plot this function using ω values that miss the singularities.
We make two plots, one with k = 1 and one with k = 2.
Also we plot a line one unit above zero.

$\omega := -19.5, -18.5 .. 55$



There will be four real roots for k = 2, but for k = 1 there will be two real and two complex roots.

Root finding

We need to find the roots of ε(k,ω) = 1-F(k,ω) = 0.
Mathcad's root finding function will do this. We must give it an initial guess.

CASE I. k = 2, four real roots $k := 2$

We will guess roots near -10, 10, 30, and 50 after looking at the figure above.

$z := -10$ $\mathrm{root}(F(k,z) - 1, z) = -10.204$

$z := 10$ $\mathrm{root}(F(k,z) - 1, z) = 10.636$

$z := 30$ $\mathrm{root}(F(k,z) - 1, z) = 29.364$

$z := 50$ $\mathrm{root}(F(k,z) - 1, z) = 50.204$

Looks like we found them all.

CASE 2. k = 1, two complex roots, two real roots $k := 1$

$z := -10$ $\mathrm{root}(F(k,z) - 1, z) = -10.582$

$z := 10$ $\mathrm{root}(F(k,z) - 1, z) = -10.582$

$z := 10$ $\mathrm{root}(F(k,z) - 1, z) = -10.582$

$z := 50$ $\mathrm{root}(F(k,z) - 1, z) = 30.582$

We know that roots often occur in complex conjugate pairs. We only found three distinct roots, so the root we have not yet found is probably the complex conjugate of the one we found near 10. We can test if the equation is solved by the complex conjugate.
In Mathcad, the complex conjugate is created by typing " (the double quote) as a modifier.

$z := 10$ This is the initial guess for the root.

$a(k) := \mathrm{root}(F(k,z) - 1, z)$

$a(k) = -10.582$ This is the complex root.

$\overline{a(k)} = -10.582$ This is the conjugate

$F(a(k), z) - 1 = 2.036 \times 10^{-3}$

$F\left(\overline{a(k)}, z\right) - 1 = 2.036 \times 10^{-3}$

Both a(k) and its complex conjugate give a value very near to zero for the value of the function, so the roots have been found.

Alternatively, we can use the complex conjugate as the initial guess for the root:

$z := 10 + 5i$ (the guess) $\mathrm{root}(F(k,z) - 1, z) = 10 + 4.859i$ (the root)

The root as a function

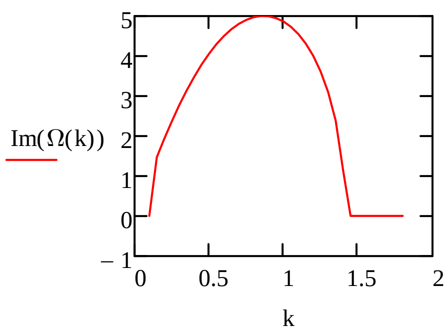
We can find the root for all values k by making the root a function of k. Call the root $\Omega(k)$.

We need to provide a guess for the location of the root: $z := 2 + 2i$

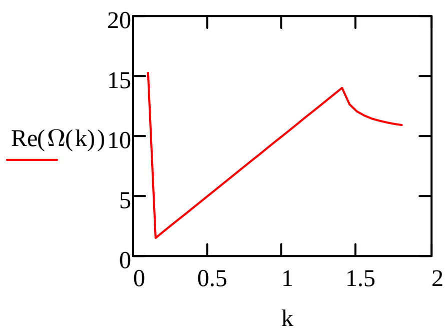
This is the function: $\Omega(k) := \mathrm{root}(F(k,z) - 1, z)$ Ω may be complex.

The growth rate of the instability is the Im part of Ω.

Define a range of k: $k := 0.05, 0.1 .. 1.8$



The most unstable wave
is one for a k value a little less than one.



This line is has the kink because the root finder has jumped from one root to another. Change the initial guess above from 2+2i to 2+3i and you will see a different "branch" of the dispersion relation.

Notes

1. To create the variable ω_p use a period between the ω and the p. This creates the two-letter name without making ω a subscripted variable.

2. The little black squares next to the first two equations indicate that they are not evaluated, and are for display only. This was done by selecting the equations, then clicking Format|Properties|Calculation|Disable Evaluation.

3. The last two curves look "OK" because the initial guess for z and the range for k was chosen carefully. If other values are used, the curve can have blank spaces where the root is not found or can have jumps when the root finder starts "zeroing in" on a different one of the four roots.

Using polyroots to find multiple roots

If we take 1-F(k,ω) and multiply through by the denominators we get the polynomial

$$G(k,\omega) := -\left(\omega^2\right)^2 + \left(2\cdot\omega^3\cdot k\cdot v\right) + \left(\omega_{pe}{}^2 + \omega_{pi}{}^2 - k^2\cdot v^2\right)\cdot\omega^2 - \left(2\cdot\omega\cdot\omega_{pi}{}^2\cdot k\cdot v\right) + k^2\cdot v^2\cdot\omega_{pi}{}^2$$

We can find all the roots of this polynomial by putting the coefficients in an array A and then using the polyroots function

We do this for $k := 1$

Each term in A(k) is one of the coefficients above.

$$A(k) := \begin{pmatrix} k^2\cdot v^2\cdot\omega_{pi}{}^2 \\ -2\cdot\omega_{pi}{}^2\cdot k\cdot v \\ \omega_{pe}{}^2 + \omega_{pi}{}^2 - k^2\cdot v^2 \\ 2\cdot k\cdot v \\ -1 \end{pmatrix} \qquad A(k) = \begin{pmatrix} 4\times 10^4 \\ -4\times 10^3 \\ -200 \\ 40 \\ -1 \end{pmatrix}$$

On the right we have printed out the numerical values for the expressions on the left side.

Now let's find the roots

$$\text{polyroots}(A(k)) = \begin{pmatrix} -10.582 \\ 10 + 4.859i \\ 10 - 4.859i \\ 30.582 \end{pmatrix}$$

The polyroots function finds the two real roots and the two complex roots.

And again for k = 2

$$\text{polyroots}(A(2)) = \begin{pmatrix} -10.204 \\ 10.636 \\ 29.364 \\ 50.204 \end{pmatrix}$$

The four roots for k = 2 are real.

Note that we did not have to write down the expression for A again because we made A a function of k.