Druyvesteyn's Probe Analysis

In 1930 Druyvesteyn showed that the electron distribution function could be found from the current-voltage characteristics of Langmuir probes. In this exercise we will apply his method to probe data from a device having two electron distributions: 1) a colder distribution (~1 eV) confined by the plasma potential, and 2) a hotter distribution (~3 eV) that is secondary electrons from the wall.
The procedure is a long one:
1) Import the data.
2) Find the plasma potential.
3) Shift the origin of the voltage scale so the plasma potential is at zero volts.
4) Find and subtract the ion current so that only the electron current is analyzed further.
5) Find the two Maxwellian distributions that best describe the electrons.
6) Use Druyvesteyn's method to find the speed distribution.
7) Use an alternate method to find the distribution function perpendicular to the probe axis.

If a plasma has a Maxwellian distribution function, the random current of electrons per unit area is: where q is the electron charge, m_e is the electron mass, n_e is the electron density and T_e is the electron temperarure in energy units.

$$J_{plasma} = n_e q \sqrt{T_e / 2\pi m_e}$$

The current to a probe of area A_p is: when the probe is at the plasma potential.

$$I_{plasma} = n_e q A_p \sqrt{T_e / 2\pi m_e}$$

Thus we can find the plasma density from:

$$n_e = \frac{I_{plasma}}{q A_p \sqrt{T_e / 2\pi m_e}}$$

Suppose, however, that the distribution is not Maxwellian. Then the density is not so easily found. Druyvesteyn [ref. 1] showed that the distribution of electron speeds is simply related to the second derivative of the probe data. He derived the following relation:

$$\rho(\varepsilon) = \frac{-4 m_e \Phi}{q^2 A_p} \frac{d^2 I}{d\Phi^2}$$

where $\rho(\varepsilon)$ is the speed distribution written as a function of energy ε, I is the probe current, Φ is the probe potential, ε is evaluated at $-q\Phi$.

The electron density is found from the speed distribution by integration:
The distribution $\rho(v)$ is defined so that the factor of $4\pi v^2$ seen in modern textbooks is contained within ρ. When $\rho(\varepsilon)$ is written, it is a shorthand notation for the awkward looking $\rho[(-2q\Phi/m_e)^{1/2}]$.

$$n = \int_0^\infty \rho(v)\, dv$$

Druyvesteyn's formula applies to disk and cylindrical Langmuir probes.
In this exercise, we will use data from a cylinder probe with dimensions:

$a := 95 \cdot 10^{-6}$ m, Probe radius

$L := 0.03$ m, Probe length

$Ap := 2 \cdot \pi \cdot a \cdot L$ Area definition

$Ap = 1.791 \times 10^{-5}$ m^2, Probe surface area

Physical constants: $m_e := 9.1094 \cdot 10^{-31}$ kg $q := 1.6 \cdot 10^{-19}$ Coulombs

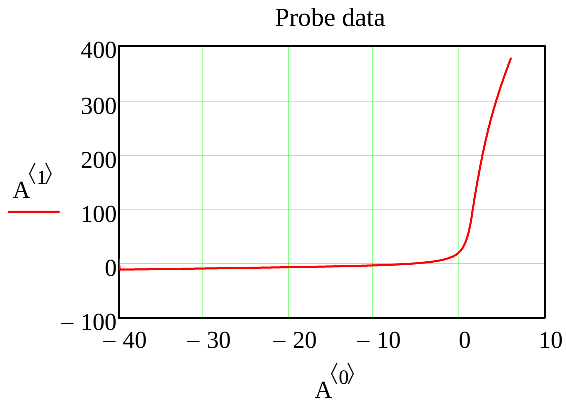
Step 1: Import the probe data:

The imported data file A below has voltage in the first column and probe current in microamps in the second column. The data points are separated by 0.1 volt and span -40 to +6 volts.

$A :=$

	0	1
0	-40	7.2
1	-39.9	-10.8
2	-39.8	-11.51
3	-39.7	-11.55
4	-39.6	-11.55
5	-39.5	-11.55
6	-39.4	-11.54
7	-39.3	-11.52
8	-39.2	...

Probe current in microamps as a function of probe voltage:



Create a data array with 3 columns: index, volts, microamps

$Jend := rows(A) - 1$ Jend is number of data points

$j := 0 .. Jend$ j is a range variable

$Index_j := j$ Index is a vector with contents j

$Data := augment(Index, A)$ column 0 is the index, col. 1 is volts, col. 2 is current

Step 2. Find the plasma potential from the derivative of the data:

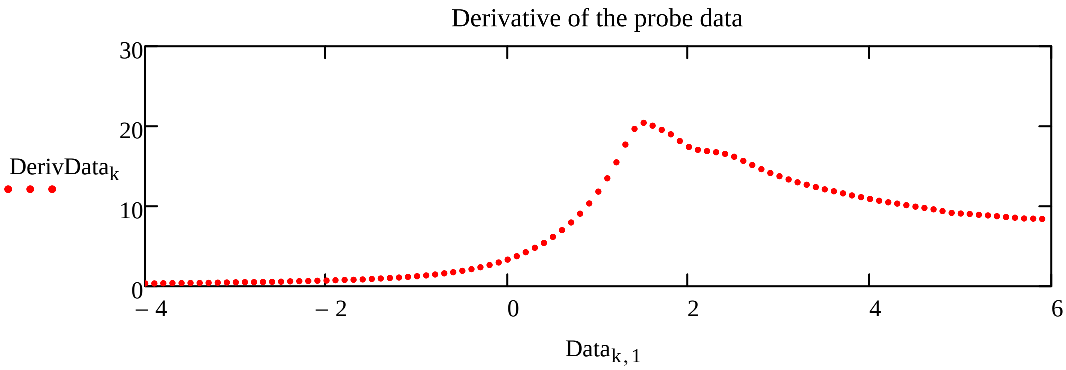
The plasma potential **Vplasma** is located at the maximum in the first derivative of the probe data.
First, we will use finite differences to find the first derivative of the data in the vicinity of zero probe voltage.

$DerivData_j := 0$ Initialize DerivData

$k := Jend - 100 .. Jend - 1$ This is the region where we might find the maximum derivative.

$DerivData_k := Data_{k+1,2} - Data_{k-1,2}$ This is the centered first derivative.

Here we have used that fact that the voltage spacing is 0.1 volts to select data from 10 volts below the end of the data to 0.1 V below the end. We can't let k go all the way to the end because the formula for DerivData uses the data at k+1.



$\max(\text{DerivData}) = 20.428$ The **max** function finds the maximum in the derivative.

The lookup function below finds the row in the data set where the maiximum is located and returns the index from column 0 of Data. In case there are multiple maxima, the subscript zero on the lookup function at the end returns the first place that the maximum is found.

$\text{Jplasma} := \text{lookup}\left[(\max(\text{DerivData})), \text{DerivData}, \text{Data}^{\langle 0 \rangle}\right]_0$ $\text{Jplasma} = 415$

Jplasma is the index for the probe voltage that is nearest to the plasma potential, Vplasma.
Next we find this probe voltage:

$\text{Vplasma} := \text{Data}_{\text{Jplasma},1}$ $\text{Vplasma} = 1.5$ Vplasma is the **trial value** of the plasma potential.

To get a better value for Vplasma, find the maximum derivative more accurately by fitting a cubic polynomial and finding the zero in the second derivative

$W := \text{submatrix}(\text{Data}, \text{Jplasma} - 2, \text{Jplasma} + 2, 1, 2)$ W is a submatrix of 5 points near the trial Vplasma.

$$W = \begin{pmatrix} 1.3 & 75.956 \\ 1.4 & 85.363 \\ 1.5 & 95.607 \\ 1.6 & 105.791 \\ 1.7 & 115.658 \end{pmatrix} \qquad \text{Fit}(V) := \begin{pmatrix} V^3 \\ V^2 \\ V \\ 1 \end{pmatrix}$$

We will fit to this subset a cubic polynomial Fit(V). The zero of second derivative is at -2b/6a, if the coefficients are a,b,c, and d.

Find the coefficients:

$$\text{Coeffs} := \text{linfit}\left(W^{\langle 0 \rangle}, W^{\langle 1 \rangle}, \text{Fit}\right) \qquad \text{Coeffs} = \begin{pmatrix} -96.167 \\ 438.893 \\ -564.452 \\ 279.284 \end{pmatrix}$$

Revised Vplasma:

$$\text{Vplasma} := \frac{-2\,\text{Coeffs}_1}{6 \cdot \text{Coeffs}_0}$$

$\text{Vplasma} = 1.521$

The plasma current at the new Vplasma can be found accurately from the fitted function. Because Coeffs and Fit are vectors, Coeffs*Fit(Vplasma) is a dot product.

$\text{Iplasma} := \text{Coeffs} \cdot \text{Fit}(\text{Vplasma})$ $\text{Iplasma} = 97.749$

Step 3. Shift the origin of the data:

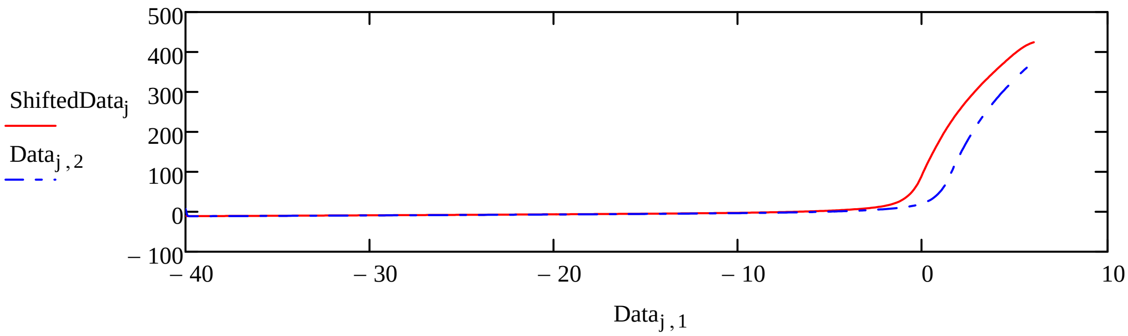
We want to have one of the probe voltages to be the plasma potential exactly, so we will shift the origin of the voltage grid so that Vplasma is on the grid. The new voltage scale is V – Vplasma. The current is found at the new set of voltages by cubic-spline interpolation.

$\text{vs} := \text{cspline}\left(\text{Data}^{\langle 1\rangle}, \text{Data}^{\langle 2\rangle}\right)$ vs is the vector with the cofficients for the spline fit.

Here we create the data set with interpolated currents

$\text{ShiftedData}_j := \text{interp}\left(\text{vs}, \text{Data}^{\langle 1\rangle}, \text{Data}^{\langle 2\rangle}, \text{Data}_{j,1} + \text{Vplasma}\right)$

Plot of probe data and shifted data



The new set of currents ShiftedData has the plasma potential at zero probe voltage.

Redefine the array Data so that it contains the shifted data:.

$\text{Data} := \text{augment}\left(\text{Data}^{\langle 0\rangle}, \text{Data}^{\langle 1\rangle}, \text{ShiftedData}\right)$ This REDEFINES the data array.

Our interpolation scheme has extrapolated the data at the positive end of the scale. Extrapolation is not likely to be accurate, but these data are not used in the analysis.

Find the plasma potential in the shifted data set by finding the index for zero probe voltage:

$\text{Jplasma} := \text{lookup}\left(0, \text{Data}^{\langle 1\rangle}, \text{Data}^{\langle 0\rangle}\right)_0$ $\text{Jplasma} = 400$

Verify that the plasma potential is where we think it is: $\text{Data}_{\text{Jplasma},1} = 0$ volts

The probe current at the plasma potential is: $\text{Data}_{\text{Jplasma},2} = 97.801$ microamps

Step 4. Find and subtract the ion current to obtain only the electron current

We will fit a polynomial to the probe current for voltages more negative than –20 V.
This voltage choice is somewhat arbitrary and there may be a better choice for other data sets.

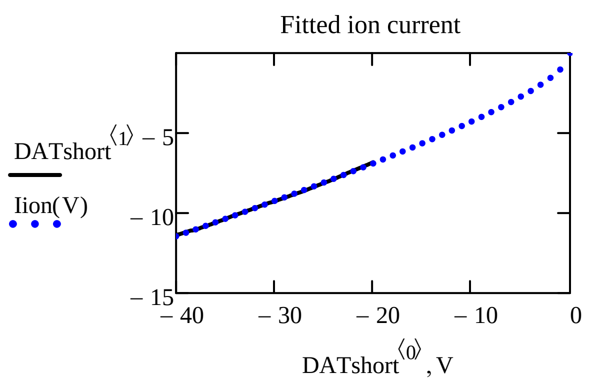
$\mathrm{DATshort} := \mathrm{submatrix}(\mathrm{Data}, 0, \mathrm{Jplasma} - 200, 1, 2)$ This creates a subset of data from the most negative voltage to –20 volts

$\mathrm{Fion}(V) := \begin{pmatrix} \sqrt{-V} \\ V \end{pmatrix}$ We fit a polynomial Fion(V) with terms proportional to V and to $(-V)^{1/2}$. The minus sign is used so that the square root is a real number.

$\mathrm{Coeff} := \mathrm{linfit}\left(\mathrm{DATshort}^{\langle 0 \rangle}, \mathrm{DATshort}^{\langle 1 \rangle}, \mathrm{Fion}\right)$ $\mathrm{Coeff} = \begin{pmatrix} -0.901 \\ 0.144 \end{pmatrix}$ These are the polynomial coefficients.

$\mathrm{Iion}(z) := \mathrm{Coeff} \cdot \mathrm{Fion}(z)$ Here we have used a dot product to multiply the coefficients with the two terms in the polynomial. z is a dummy variable.

$V := -40, -39 .. 0$ Create a range of voltages to put in the function Iion.



The points are the fitted ion current from –40 V to 0 V. The line is the data used for the fit. We did not use more positive voltages for the fit because electrons are collected at voltages more positive than –20. This voltage value depends upon the plasma being studied. The function Iion(V) allows us to extrapolate the ion current to the region –20 V to 0 V.

Subtract the ion current from the data to get the electron current

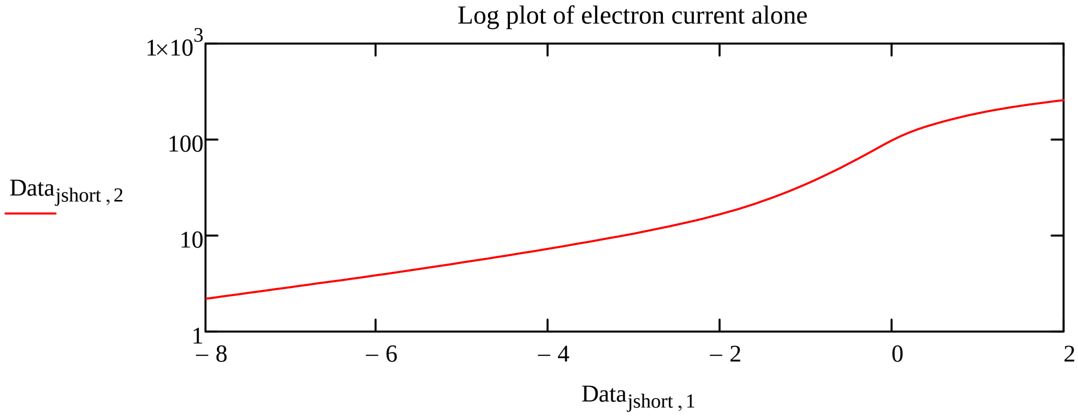
$\mathrm{Iion}(z) := \mathrm{if}(z \le 0, \mathrm{Iion}(z), 0)$ Redefine ion current function to be zero for positive voltages, because no ions are collected at positive voltage.

$\mathrm{IonC}_j := \mathrm{Iion}\left(\mathrm{Data}_{j,1}\right)$ The vector IonC contains the fitted ion current.

$\mathrm{Data}^{\langle 2 \rangle} := \mathrm{Data}^{\langle 2 \rangle} - \mathrm{IonC}$ Redefine the second column of Data so that it contains the electron current alone, obtained by subtracting the ion current from the total current.

Plot the electron current:

$\text{jshort} := \text{Jplasma} - 80 .. \text{Jplasma} + 20$ jshort limits the plot so that it is from -8 V to +2 V



Note that the electron current has a lower slope from –8 to –2 volts and a larger slope from –1 volt to 0 volts. This indicates that the distribution is non-Maxwellian.

The plasma device used for this data has a colder plasma with temperature near 1 eV and a hotter plasma of secondary electrons emitted by the wall with 3 eV temperature.

Step 5A. Perform a preliminary simple probe data analysis to find n_e and T_e
Use the data from 0 V to –1 V and fit a Maxwellian.

Create a subset of data, from –1 V to 0 V:

$\text{DAT} := \text{submatrix}(\text{Data}, \text{Jplasma} - 10, \text{Jplasma}, 0, 2)$

This subset of data is chosen because previous work with data from this device has shown a temperature near 1 eV.

The electron temperature is the inverse of the slope of a log plot of the electron current.
Use **slope** and **intercept** to find the line that fits best:

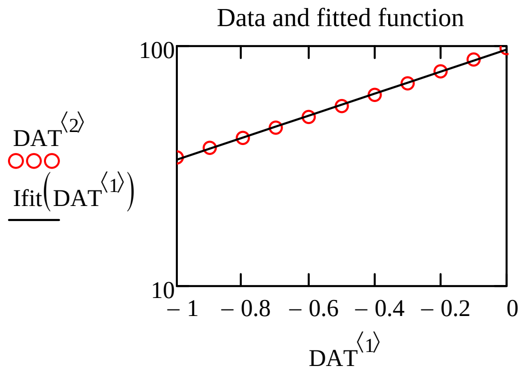
$\text{logslope} := \text{slope}\left(\text{DAT}^{\langle 1\rangle}, \overrightarrow{\ln\left(\text{DAT}^{\langle 2\rangle}\right)}\right)$ $\text{logintercept} := \text{intercept}\left(\text{DAT}^{\langle 1\rangle}, \overrightarrow{\ln\left(\text{DAT}^{\langle 2\rangle}\right)}\right)$

Above we used the vectorize function to obtain the vector of logarithms.

Preliminary T_e value is: $T := \dfrac{1}{\text{logslope}}$ $T = 0.948$ eV

From the fitted slope and intercept we can create the fitted function: $\text{Ifit}(V) := e^{(\text{logintercept} + \text{logslope} \cdot V)}$

This plot provides a check on the fitted Maxwellian:



The fit looks accurate.

The density n_e is found from the current I_{plasma} at the plasma potential using:

$$n_e = \frac{I_{plasma}}{qA_p\sqrt{T_e / 2\pi m_e}}$$

The preliminary n_e value is:

$$ne1 := \frac{\left(Iplasma \cdot 10^{-6}\right)}{\sqrt{\frac{q \cdot T}{2 \cdot \pi \cdot m_e}} \cdot Ap \cdot q}$$

$ne1 = 2.096 \times 10^{14}$ m^{-3}

Note that qT has the proper energy units for temperature T in eV.

Step 5B. Fit two Maxwellian distributions to the probe current

Our data is fit better by a sum of two Maxwellian distributions. We will fit a **bi-Maxwellian** and later show that this distribution is approximately lthe same as the distribution from Druyvesteyn's method.

$$BiMax(V, Isat1, Te1, Isat2, Te2) := Isat1 \cdot \exp\left(\frac{V}{Te1}\right) + Isat2 \cdot \exp\left(\frac{V}{Te2}\right)$$

This is the bi-Maxwellian function that will be fit to the data.

Isat2 is the "saturation current" (the current at the plasma potential) for the high energy tail and Te2 is the temperature of the high energy tail. From our simpler analysis using a single Maxwellian, we can start with the initial guesses:

$Isat1 := Iplasma$ I is in microamps. $Te1 := T$ T is in electron volts.

As a first guess for the "tail" parameters we will try: $Isat2 := \frac{Isat1}{10}$ $Te2 := 3Te1$

Use the minimize function to find the best fit:

The **minimize function** will find the values of variables that minimize the value of a function. We will define an error function Err(Isat1,Te1,Isat2,Te2) that is simply related to the difference between the fitted function, F(V,Isat1,Te1,Isat2,Te2), and the data. The error is measured by summing the squares of the differences, hence we are implementing a "least squares" method.

$k := Jplasma - 80 .. Jplasma$ This is the range of k for which the function must fit the data.

$IDAT := Data^{\langle 2 \rangle}$ Vector containing the electron current data

$\Phi := Data^{\langle 1 \rangle}$ Vector containing the probe voltage Φ.
–Φ will be used for the **electron energy in eV**.

The function Err is the sum of the squares of the errors:

$$\mathrm{Err}(\mathrm{Isat1},\mathrm{Te1},\mathrm{Isat2},\mathrm{Te2}) := \sum_k \left(\mathrm{IDAT}_k - \mathrm{BiMax}\left(\Phi_k,\mathrm{Isat1},\mathrm{Te1},\mathrm{Isat2},\mathrm{Te2}\right)\right)^2$$

$$P := \mathrm{Minimize}(\mathrm{Err},\mathrm{Isat1},\mathrm{Te1},\mathrm{Isat2},\mathrm{Te2})$$

The **minimize** function finds the best values for Isat, Isat2, Te, and Te2. This function, unlike linfit, requires initial guesses for the variables.

$$P = \begin{pmatrix} 75.798 \\ 0.683 \\ 22.681 \\ 3.412 \end{pmatrix} \begin{matrix} \mathrm{Isat1} \\ \mathrm{Te1} \\ \mathrm{Isat2} \\ \mathrm{Te2} \end{matrix}$$

The vector P that is returned by the minimize function gives us the values for the parameters that best fit the data.

When two Maxwellians are used, the temperature of the colder Maxwellian, 0.683 eV, is somewhat lower than the 0.948 eV that we found using a single Maxwellian. The two-Maxwellian method is decribed in more detail in ref. 2.

Define the bi-Maxwellian distribution function (to be used later):

We will use the function that fit the current data to construct a bi-Maxwellian distribution function.

$\mathrm{Isat1} := P_0$ $\mathrm{Tcold} := P_1$ $\mathrm{Isat2} := P_2$ $\mathrm{Thot} := P_3$

Here we find the densities, neCold and neHot, of the hot and cold parts of the distribution:

$$\mathrm{neCold} := \frac{\left(\mathrm{Isat1}\cdot 10^{-6}\right)}{\sqrt{\dfrac{q\cdot \mathrm{Tcold}}{2\cdot\pi\cdot m_e}}\cdot \mathrm{Ap}\cdot q} \qquad \mathrm{neHot} := \frac{\left(\mathrm{Isat2}\cdot 10^{-6}\right)}{\sqrt{\dfrac{q\cdot \mathrm{Thot}}{2\cdot\pi\cdot m_e}}\cdot \mathrm{Ap}\cdot q}$$

$\mathrm{neCold} = 1.915 \times 10^{14}$ m^{-3} $\mathrm{neHot} = 2.563 \times 10^{13}$ m^{-3}

$\mathrm{Tcold} = 0.683$ eV $\mathrm{Thot} = 3.412$ eV

Construct the speed distribution:

$k := \mathrm{Jplasma} - 80\,..\,\mathrm{Jplasma} - 1$ Prevent singularity at Jplasma.

$$\rho\mathrm{bimax}_k := \left[\mathrm{neCold}\cdot\left(\frac{m_e}{2\cdot\pi\cdot q\cdot \mathrm{Tcold}}\right)^{\frac{3}{2}}\cdot e^{\frac{\Phi_k}{\mathrm{Tcold}}} + \mathrm{neHot}\cdot\left(\frac{m_e}{2\cdot\pi\cdot q\cdot \mathrm{Thot}}\right)^{\frac{3}{2}}\cdot e^{\frac{\Phi_k}{\mathrm{Thot}}}\right]\cdot\left[4\cdot\pi\cdot\left(-2\cdot\frac{q}{m_e}\cdot\Phi_k\right)\right]$$

The last factor above is the same as $4\pi v^2$.

Step 6: Use Druyvesteyn's method to find the speed distribution ρ and n_e

$\mathrm{IDAT} := \mathrm{Data}^{\langle 2 \rangle}$ Vector containing the electron current data

$\Phi := \mathrm{Data}^{\langle 1 \rangle}$ Vector containing the probe voltage. $-\Phi$ will be used for the **electron energy in eV**.

$\Delta V := A_{2,0} - A_{1,0}$ $\Delta V = 0.1$ Spacing of the voltage of the probe data points

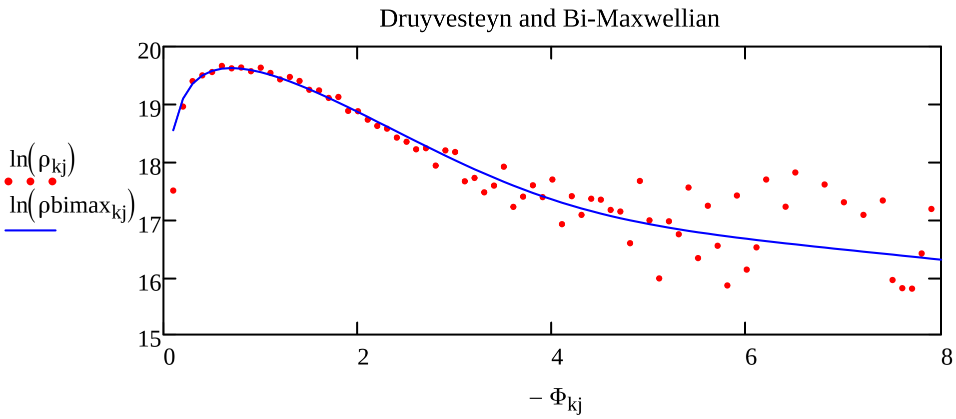
$kj := \mathrm{Jplasma} - 80 .. \mathrm{Jplasma}$ Index that creates a subset of the data starting 8 V below the end.

Druyvesteyn 's formula

$$\rho_{kj} := \frac{-4 \cdot m_e \cdot \Phi_{kj}}{10^6 \cdot q^2 \cdot \mathrm{Ap}} \cdot \left[\frac{\mathrm{IDAT}_{kj+1} - 2 \cdot \mathrm{IDAT}_{kj} + \mathrm{IDAT}_{kj-1}}{(\Delta V)^2} \right]$$

This is Druyvesteyn's formula (from page 1) using the finite difference form of the second derivative.

Speed distribution from Druyvesteyn's method (points) and the bi-Maxwellian (line):



Integrate the distribution ρ to get the density

We must evaluate:: $n = \int_0^\infty \rho(v)\, dv$ but ρ is defined at equally spaced points in energy.

Note that: $d\varepsilon = m_e v dv$ which implies $dv = \frac{d\varepsilon}{m_e v} = \frac{d\varepsilon}{\sqrt{-2 m_e q \Phi}}$

The density integral in finite-difference form becomes a sum: $$n = \int_0^\infty \frac{\rho(\varepsilon)\, d\varepsilon}{\sqrt{-2 m_e q \Phi}} = \sum_k \frac{\rho(q\Phi_k)\, q\Delta V}{\sqrt{-2 m_e q \Phi_k}}$$

The sum over k starts at a very negative probe voltage (-8 V for example) and ends at the plasma potential where $\Phi = 0$. This voltage is omitted from the sum to avoid the singularity. Note that ρ goes also to zero at $\Phi = 0$, so the final term in the sum is zero by L'Hopital's rule.

Comparison of density from the simple analysis to density from Druyvesteyn's analysis:

This is the integral for the density:

$$ne2 := \sum_{kj = Jplasma-80}^{Jplasma-1} \left[\frac{\rho_{kj}}{\sqrt{-2 \cdot m_e \cdot q \cdot (\Phi_{kj})}} \cdot q \cdot \Delta V \right]$$

Druyvesteyn's density:

$ne2 = 2.033 \times 10^{14}$ m^{-3}

Compare this value to the value found in Step 5A: $ne1 = 2.096 \times 10^{14}$ m^{-3}

Make a plot of $\rho/4\pi v^2$ to compare with the fitted Maxwellian:

Define a new function F(ε) that is $\rho/4\pi v^2$ with $\varepsilon = -2q\Phi/m_e$.
This function can be compared with the Maxwellian.
Note that $v^2 = -2q\Phi/m_e$.

$$F_{kj} := \frac{\rho_{kj}}{4 \cdot \pi \cdot \left(-2 \cdot \frac{q}{m_e} \cdot \Phi_{kj}\right)}$$

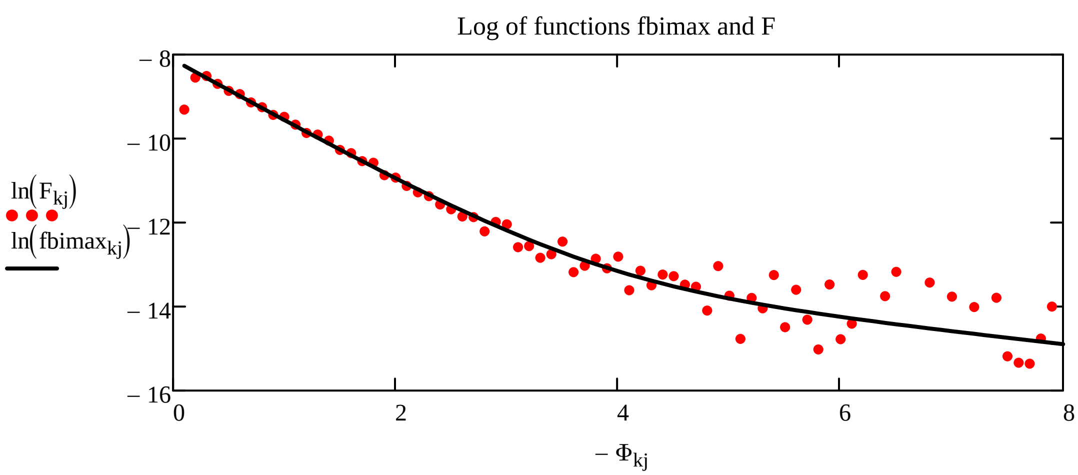
Recall that a simple Maxwellian distribution function is:
if ε is kinetic energy and T_e is in energy units.
A bi-Maxwellian is the sum of two of these:

$$f(\varepsilon) = \left(\frac{n_e}{2\pi T_e}\right)^{\frac{3}{2}} \exp(-\varepsilon / T_e)$$

$$fbimax_k := \left[neCold \cdot \left(\frac{m_e}{2 \cdot \pi \cdot q \cdot Tcold}\right)^{\frac{3}{2}} \cdot e^{\frac{\Phi_k}{Tcold}} + neHot \cdot \left(\frac{m_e}{2 \cdot \pi \cdot q \cdot Thot}\right)^{\frac{3}{2}} \cdot e^{\frac{\Phi_k}{Thot}} \right]$$

Note that qT is the temperature in energy units because T is in eV.
Φ/T is dimensionless when T is in eV.

Comparison of the Maxwellian from the bi-Maxwellian analysis (solid line) to the distribution from the Druyvesteyn analysis (points).



We have used the ln function above rather than a log plot because some values are negative because of noise and would not plot. f is from the simple analysis and F is from Druyvesteyn's analysis.

Comparison of temperatures from the two methods:

We will find a temperature from the slope of the distribution function. First create a subset of F and Φ:

$\text{Fsub} := \text{submatrix}(F, \text{Jplasma} - 3, \text{Jplasma} - 20, 0, 0)$

$\Phi\text{sub} := \text{submatrix}(\Phi, \text{Jplasma} - 3, \text{Jplasma} - 20, 0, 0)$

This subset omits two "bad" points near the origin and goes to –2 volts.

Find the slope of a log plot:

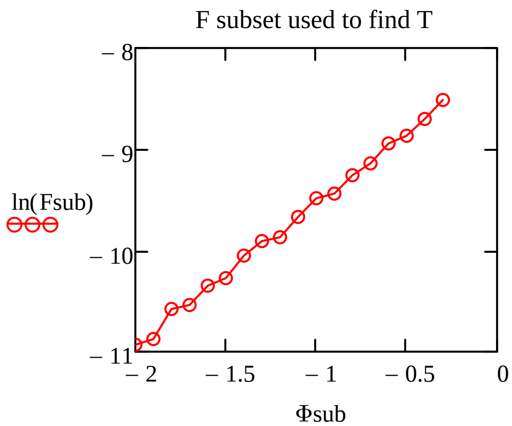
$\text{logslope} := \text{slope}\left(\Phi\text{sub}, \overrightarrow{\ln(\text{Fsub})}\right)$

T is the inverse of the slope.

$\text{TDruy} := \dfrac{1}{\text{logslope}}$ $\quad \text{TDruy} = 0.712$ eV

This temperature is near the 0.683 eV from the bi-Maxwellian fit but it is lower than the temperature from the simple Maxwellian:

$T = 0.948$ eV



Step 7. An alternate analysis that finds F(vperp):

From cylindrical probe data, we can also find the distribution function projected onto the plane perpendicular to the axis of the probe, defined as:

$$F(v_{perp}) = \int_{-\infty}^{\infty} f(v_{perp}, v_z)\, dv_z$$

The probe data are also related to the distribution function F(vperp) that is the distribution function projected into the plane perpendicular to the probe axis. This relationship and the inverse relationship that allows the distribution to the calculated from the data are difficult to derive (see reference 4). We will simply make use of the result.

Based on our earlier simple analysis, this two-dimensinal distribution is approximately: Fperp is normalized for two velocity dimensions and will be compared to F(vperp) from our analysis.

$$\text{Fperp}(\varepsilon) := \text{ne1} \cdot \left(\frac{m_e}{2 \cdot \pi \cdot q \cdot T}\right) \cdot e^{\frac{-\varepsilon}{T}}$$

The values of F(ε) for $\varepsilon = -q\Phi_k$ are found iteratively.

$\text{kmin} := \text{Jplasma} - 100$ kmin is the index for the negative probe voltage where the anaysis begins.

$F_{kmin} := 0$ The distribution is assumed zero at the most negative voltage.

$$F_{kmin+1} := \frac{\dfrac{10^{-6} \cdot IDAT_{kmin+1} \cdot m_e}{2 \cdot Ap \cdot q} \cdot \sqrt{\dfrac{m_e}{2}}}{\dfrac{2}{3} \cdot \left(q \cdot \Phi_{kmin+1} - q \cdot \Phi_{kmin}\right)^{\frac{3}{2}}}$$

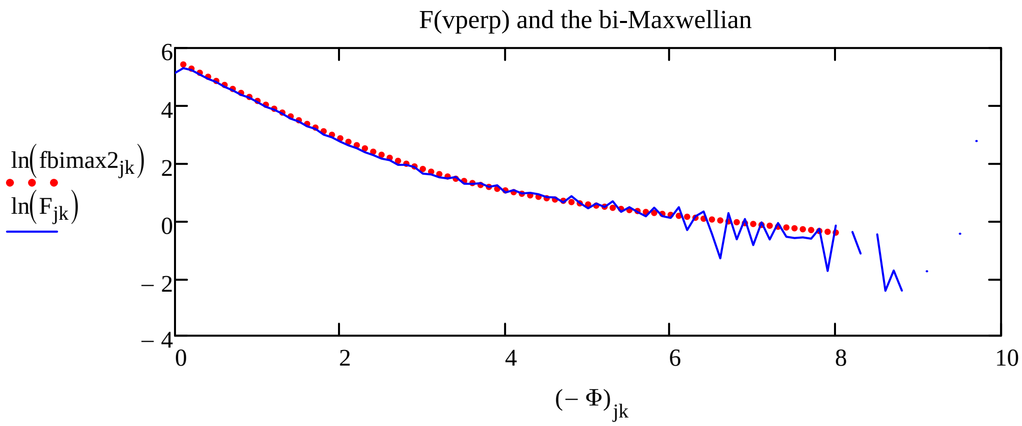
This defines the distribution at the next energy value.

$jk := kmin + 2 .. Jplasma$ The next line defines F at all subsequent energy values (see ref. 4).

$$F_{jk} := \frac{\dfrac{IDAT_{jk} \cdot m_e}{2 \cdot q \cdot Ap \cdot 10^6} \cdot \sqrt{\dfrac{m_e}{2}} - \left[\displaystyle\sum_{jj = kmin}^{jk-1} \left[F_{jj} \cdot \left[\frac{2}{3} \cdot \left(q \cdot \Phi_{jk} - q \cdot \Phi_{jj-1}\right)^{\frac{3}{2}} - \frac{2}{3} \cdot \left(q \cdot \Phi_{jk} - q \cdot \Phi_{jj}\right)^{\frac{3}{2}}\right]\right]\right]}{\dfrac{2}{3} \cdot \left(q \cdot \Phi_{jk} - q \cdot \Phi_{jk-1}\right)^{\frac{3}{2}}}$$

Using the bi-Maxwellian analysis, we can construct the distribution that we expect.
This is a two dimensional distribution, so the 2/2 power appears in the normalization,
otherwise it is like the fbimax defined on p. '0

$$fbimax2_k := \left[neCold \cdot \left(\frac{m_e}{2 \cdot \pi \cdot q \cdot Tcold}\right)^{\frac{2}{2}} \cdot e^{\frac{\Phi_k}{Tcold}} + neHot \cdot \left(\frac{m_e}{2 \cdot \pi \cdot q \cdot Thot}\right)^{\frac{2}{2}} \cdot e^{\frac{\Phi_k}{Thot}}\right]$$



Some of the points are missing because the noise makes the distribution function negative, and the log of negative values will not appear on the plot.

Note that this distribution (points) is much less noisy than Druyvesteyn's speed distribution (p. 9) and is similar to the two-dimensional distribution Fperp(ε) based on earlier values for n and T. The points at the most negative probe voltages are not accurate because the sum does not start at an infinitely negative voltage. The noise in this distribution is less evident than the noise in Druyvesteyn's distribution.

The density is found from: $n = \int_{-\infty}^{\infty} F(v_{perp}) 2\pi v_{perp} dv_{perp}$ Note that $d\varepsilon = m v_{perp} dv_{perp}$.

In finite-difference form n becomes the sum: $ne3 := \frac{2 \cdot \pi \cdot q \cdot \Delta V}{m_e} \cdot \sum_{kj = kmin}^{Jplasma-1} \left(\frac{F_{kj} + F_{kj+1}}{2} \right)$ $ne3 = 1.996 \times 10^{14}$ m^{-3}

Recall that: $ne1 = 2.096 \times 10^{14}$ $ne2 = 2.033 \times 10^{14}$ The new n is very near the old n values.

What is the temperature from the slope of this distribution?

Next, we will find the temperature from the slope of a subset of the distribution.

$Fsub := submatrix(F, Jplasma - 3, Jplasma - 20, 0, 0)$

$\Phi sub := submatrix(\Phi, Jplasma - 3, Jplasma - 20, 0, 0)$

This data subset omits 3 "bad" points near the origin and goes to –2 volts.

Find T from the slope of a log plot: $logslope := slope\left(\Phi sub, \overrightarrow{\ln(Fsub)}\right)$ $logslope = 1.368$

$logintercept := intercept\left(\Phi sub, \overrightarrow{\ln(Fsub)}\right)$ $FitF(V) := e^{logintercept + logslope \cdot V}$

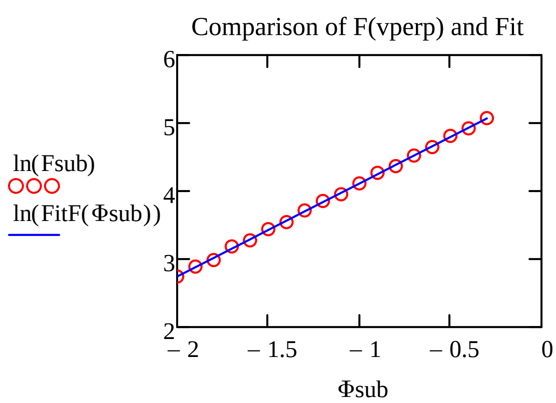
$T3 := \frac{1}{logslope}$ $T3 = 0.731$ eV

From the bi-Maxwellian fit, we got:

$P_1 = 0.683$ eV

And from Druvesteyn's analysis we got:

$TDruy = 0.712$ eV



The three different values obtained for temperature are in reasonable agreement.

Alternate Step 5. Druyvesteyn's analysis with smoothing

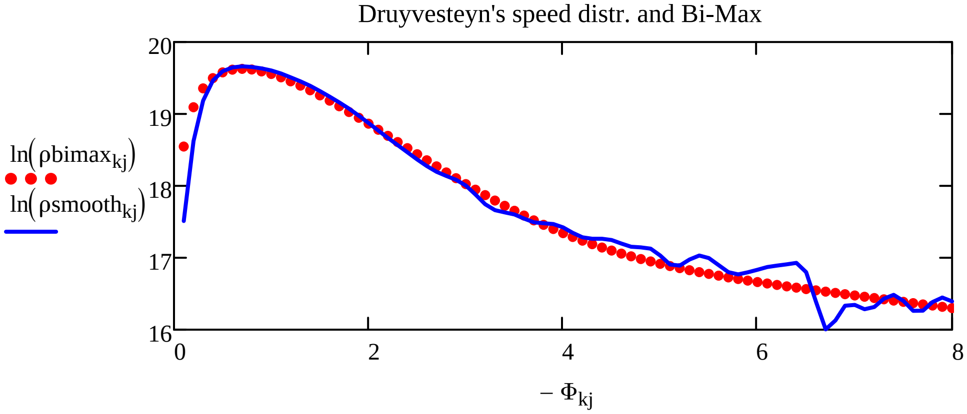
This section implements the method of Savitzky and Golay in ref. 3.

$IDAT := ksmooth(\Phi, IDAT, 0.4)$

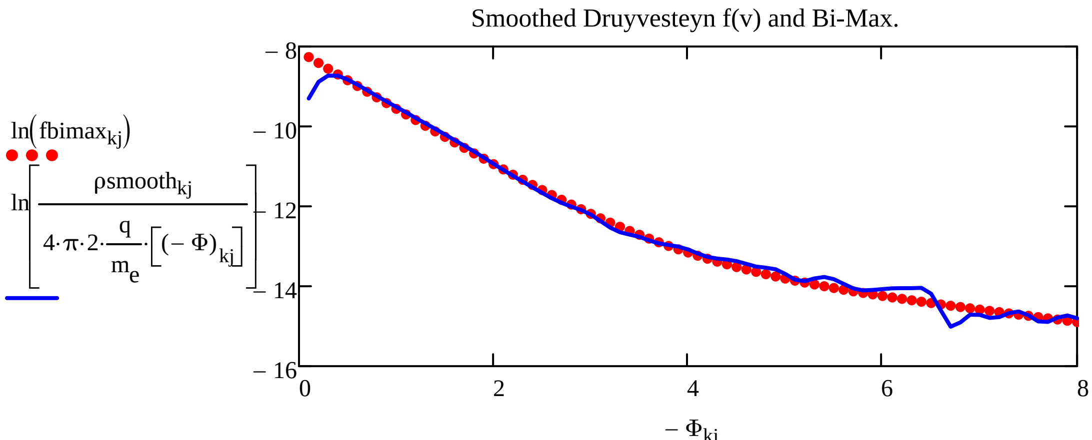
The **ksmooth function** smooths the IDAT points.
The width of the smoothing function is set to 0.4 volts.
This value, found by trial and error, smooths the function without sacrificing too much detail.

$$\rho smooth_{kj} := \frac{-4 \cdot m_e \cdot \Phi_{kj}}{10^6 \cdot q^2 \cdot Ap} \cdot \left[\frac{IDAT_{kj+1} - 2 \cdot IDAT_{kj} + IDAT_{kj-1}}{(\Delta V)^2}\right]$$

This is Druyvesteyn's formula using smoothed data.



The smoothed distribution looks "nicer."
We can make the same comparison with the distribution function rather than the speed distribution.



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